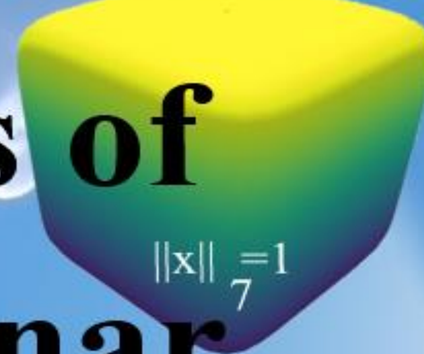
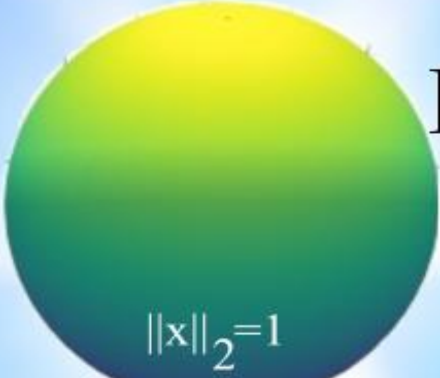


Proceedings of the 7th Seminar on Functional Analysis and its Applications

1-2 March, 2023



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IMAM KHOMEINI INTERNATIONAL UNIVERSITY

The 7th Seminar on Functional Analysis and its Applications

1-2 March 2023, Qazvin, IKIU, IRAN

Book of Abstracts

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The 7th Seminar on Functional Analysis and its Applications

Day 1: Wednesday 1 March 2023

Tehran Local Time				
8:30- 9:30	Opening Ceremony 1. Moments with the Holy Quran 2. National Anthem of the Islamic Republic of Iran 3. A short clip about Imam Khomeini International University 4. Speech by the Vice President for Research and Technology 5. Speech by the President of the Iranian Mathematical Society (IMS) 6. Conference Chair Report			
9:30-9:40	T. Break			
	Invited Speaker		Title	
9:40-10:30	J. B. Seoane-Sepulveda I10940		Linearity in nonlinear settings	
10:30-10:40	T. Break			
	Contributed Talks			
	Session A Real and Functional Analysis	**Session B** Complex Analysis and Operator Theory	**Session C** Harmonic and Nonlinear Analysis and Banach Algebras	**Session D** Differential Equations and Applied Functional Analysis
10:40-11:00	M. Gabeleh A11040	M. Amani B11040	A.H. Sanatpour C11040	R. Saadati 1 D11040
11:00-11:20	V. Keshavarz A11100	A. Babaei 1 B11100	M. Asadipour C11100	A. Khaleghi D11100
11:20-11:40	M. Hassani 1 A11120	A. Barani B11120	R. Eskandari C11120	S. Hajiaghasi 1 D11120
11:40-14:00	Lunch Break			
14:00-14:20	M. Asadi A11400	M. Keshtkar 2 B11400	M. Moosavi C11400	S. Shakeri D11400
14:20-14:40	B. Mohammadi 1 A11420	S. Eskandari B11420	Z. Kefayati C11420	T. Kasbi Gharahasanlou 1 D11420
14:40-15:00	M. R. Karimzadeh A11440	M. Salehi B11440	E. Yazdan C11440	M. Shahriari 1 D11440
15:00-15:10	T. Break			
	Invited Speaker		Title	
15:10-16:00	**J. Mashreghi** I11510		Taylor polynomials on local Dirichlet spaces	
16:00-16:10	T. Break			
	Contributed Talks			
	Session A Real and Functional Analysis	**Session B** Complex Analysis and Operator Theory	**Session C** Harmonic and Nonlinear Analysis and Banach Algebras	**Session D** Differential Equations and Applied Functional Analysis
16:10-16:30	F. Mirdamadi 1 A11610	M. Djahangiri 1 B11610	M. R. Jabbarzadeh C11610	R. Saadati 2 D11610
16:30-16:50	S. Fatemi A11630	R. Parvinianzadeh B11630	M. Kian C11630	M. Taghavi D11630
16:50-17:10	P. Heiatian Naeini 1 A11650	H. Rahmatan B11650	J. Koushki C11650	M. Mirzapour 1 D11650
	End of Day 1			

Day 2: Thursday 2 March 2023

Tehran Local Time				
	Invited Speaker		**Title**	
08:30-9:20	**J. Yong** I20830		Noncommutative good-λ inequalities	
09:20-09:30	T. Break			
Contributed Talks				
	Session A Real and Functional Analysis	**Session B** Complex Analysis and Operator Theory	**Session C** Harmonic and Nonlinear Analysis and Banach Algebras	**Session D** Differential Equations and Applied Functional Analysis
09:30-09:50	M. T. Heydari A20930	A. Babaei 2 B20930	A. Ghaffari C20930	M. Latifi D20930
09:50-10:10	R. Rezavand A20950	M. Djahangiri 2 B20950	S. M. Manjegani C20950	T. Norouzi Ghara D20950
10:10-10:30	F. Mirdamadi 2 A21010	M. Keshtkar 1 B21010	M. Saadati C21010	M. Shahriari 2 D21010
10:30-10:40	T. Break			
10:40-11:00	M. Hassani 2 A21040	Z. Saeidikia B21040	A. Shekari C21040	S. Hajiaghasi 2 D21040
11:00-11:20	A. Safari Hafshejani 2 A21100	L. Nasiri B21100	A. Safari Hafshejani 1 C21100	T. Kasbi Gharahasanlou 2 D21100
11:20-11:40	P. Heiatian Naeini 2 A21120	M. Rostamian Delavar 1 B21120	B. Taherkhani C21120	M. Mirzapour 2 D21120
11:40-14:00	Lunch Break			
	Invited Speaker		**Title**	
14:00-14:50	**A. Aleman** I21400		Generalizations of de Branges-Rovnyak spaces	
14:50-15:00	T. Break			
15:00-15:20	B. Mohammadi 2 A21500	M. Khaleghi B21500	A. Zivari-Kazempour C21500	T. Soltani D21500
15:20-15:40	M. Salehnejad A21520	M. Rostamian Delavar 2 B21520	S. H. Sayedain Boroujeni C21520	A. Sobhani D21520
15:40-16:00	H. Lakzian A21540	J. Shaffaf B21540		F. Tosnejad D21540
16:00-16:10	T. Break			
	Invited Speaker		Title	
16:10-17:00	K. Zhu I21610		Sub-Bergman Hilbert spaces in the unit disk	
17:00-17:20	Closing Ceremony			

Dear Colleagues and Friends,

It is our pleasure to welcome you to the **7th Seminar on Functional Analysis and its Applications**, held during March 1-2, 2023 at Department of Pure Mathematics, Imam Khomeini International University, Qazvin, Iran. The Seminar on Functional Analysis and its Applications is a national seminars initiated by the Iranian Mathematical Society in 1986. The first seminar was hosted by Sharif University of Technology in 1986, and the latest one by the University of Isfahan in 2020.

We wish the seminar to become a platform for raising discussions and creating joint research projects among the participants. Planning a virtual seminar has the advantage that more distinguished mathematicians from around the globe may participate in the event. Fortunately, this goal was achieved, and 5 eminent figures form China, the United States, Canada, Spain, and Sweden have accepted our invitation to deliver invited talks in a variety of subjects.

We are happy to announce that the seminar was well received by mathematicians of our country. We have got more than 100 papers; but the time slot was limited, so that by the recommendation of our prominent scientific committee members we were forced to be more selective to accept just 78 papers for contributed talks. Needless to say that some of the accepted papers were transferred to poster presentation section; this does not necessarily mean that the papers lack the quality requirements of the scientific committee.

The seminar's program includes 5 invited talks, 74 contributed talks, and 4 poster presentations. We have assigned a 6-digit alphanumeric code to every contributed talk; the first character is A, B, C, or D, followed by a 5-digit number. These letters stand for Sections A, B, C, or D. The subsequent number following the letter is either 1 or 2, which indicates that the talk will be presented on Day 1 or Day 2 of the seminar. The last four-digit number represents the time of presentation.

For example, the code

A21430 means that the talk will be presented in **Section A, on Day 2, at 14:30**.

A lecture whose code begins with "**A**" will takes place at room **A** of

https://www.skyroom.online/ch/mathikiu/rooma/l/en

The same applies to the letters B, C, and D.

We wish to express our sincere thanks to our invited speakers and all participants for sharing their latest findings in this seminar. Last, but not least, we record our gratitude to our scientific committee members, twenty reputed functional analysts from across the country, for their invaluable efforts in evaluating the received papers in a reasonable amount of time. Their prompt action and accuracy of decision is greatly appreciated. We also record our thanks to our executive committee at IKIU for their cooperation and help; they have done everything to make this event pleasant.

Finally, we wish you every success in the future, and hope that you all will enjoy this event.

Ali Abkar (Chair of Seminar),

Abdolrahman Razani (Chair of Scientific Committee),

Morteza Oveisiha (Chair of Executive Committee)

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Soghra Sharifi (Faculty Staff)

During the seminar, the following people helped us in different capacities. Some helped us in organizing and managing the sessions; some distinguished colleagues agreed to become the chair of our sessions; and finally some PhD students helped us with playing the recorded videos, and so on. We wish to record our sincere thanks to all of them:

Chairmanship of speakers' sessions

Farshid Abdollahi, Shiraz University
Ali Armandnejad, Vali-e-Asr University of Rafsanjan
Ali Farajzadeh, Razi University
Morteza Fotouhi, Sharif University of Technology
Moosa Gabeleh, Ayatollah Borujerdi University
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PhD students

Yazdan Bayat
Seyedeh Atefeh Falah Shams
Sakineh Hajiaghasi
Zohreh Kefayati
Ali Khaleghi
Zahra Saeidikia
Tahreh Soltani

CONTENTS

POSTERS

مجموعه مقاله های فارسی

Invited Talks

Generalizations of de Branges-Rovnyak Spaces

Alexandru Aleman

University of Lund, Sweden
Email: alexandru.aleman@math.lu.se

ABSTRACT

Some natural generalizations of sub-Hardy (de Branges-Rovnyak) spaces are Hilbert spaces of analytic functions in the disc, where the backward shift acts as a contraction. The sub-Bergman spaces introduced by K. Zhou area different generalization which is interesting in its own right. These are essentially a particular case of Hilbert spaces of analytic functions in the disc, where the forward shift satisfies a famous hereditary inequality of S.Shimorin. The basic observation used in the talk is that such spaces are reproducing kernel Hilbert spaces whose kernel is obtained by dividing a given kernel (like the Szegö or Bergman kernel) by a normalized completeNevanlinna-Pick kernel. The aim is to deduce some general properties of these objects. We derive a useful formula for the norm and discuss some approximation results as well as some embedding theorems. This is a report about recent joint work with F. Weistr¨om Dahlin as well as previous work joint with B. Malman.

Taylor polynomials on Local Dirichlet Spaces

Javad Mashreghi

Dèpartement de mathèmatiques et de statistique, Universitè Laval, Quèbec, QC, Canada G1K 7P4
Email: Javad.Mashreghi@mat.ulaval.ca

ABSTRACT

The partial Taylor sums S_n, $n \geq 0$, are finite rank operators on any Banach space of analytic functions on the open unit disc. In the classical setting of disc algebra $\mathcal{A}$, the precise value of $\|S_n\|_{\mathcal{A}\to\mathcal{A}}$ is not known. These numbers are referred as the Lebesgue constants and they grow like $\log n$, modulo a multiplicative constant, when n tends to infinity. We study $\|S_n\|$ when it acts on the local Dirichlet space $\mathcal{D}_\zeta$. There are several distinguished ways to put a norm on $\mathcal{D}_\zeta$ and each choice naturally leads to a different operator norm for S_n, as an operator on $\mathcal{D}_\zeta$. We consider three different norms on $\mathcal{D}_\zeta$ and, in each case, evaluate the precise value of $\|S_n\|_{\mathcal{D}_\zeta\to\mathcal{D}_\zeta}$. In each case, we also show that the maximizing function is unique.

Linearity in Nonlinear Settings

Juan B. Seoane-Sepúlveda

Department of Applied Mathematics and Mathematical Analysis, School of Mathematical Sciences, Complutense University of Madrid, Madrid, Spain
Email: jseoane@ucm.es

ABSTRACT

For the last decade there has been a generalized trend in Mathematics on the search for large algebraic structures (linear spaces, closed subspaces, or infinitely generated algebras) composed of mathematical objects enjoying certain special properties. One of the earliest results in this directions was a famous theorem by V. I. Gurariy (1966), in which he showed that the set of Weierstrass' monsters (continuous nowhere differentiable functions)contains (up to the zero function) an infinite dimensional linear space. This trend has caught the eye of many researchers and has also had a remarkable influence in Real and Complex Analysis, Set Theory, Operator Theory, Summability Theory,Polynomials in Banach spaces, Hypercyclicity and Chaos, Axiomatic Set Theory, Probability Theory, and general Functional Analysis.

Throughout this lecture we shall present an account on the advances and on the state of the art of this trend, nowadays known as lineability and spaceability. Open problems and questions will also be provided throughout the talk. On top of that we shall also discuss new potential directions of research and techniques to tackle some of these problems.

Noncommutative good-λ inequalities

Jiao Yong

School of Mathematics and Statistics Central South University Changsha, Hunan, 410083 P.R.China
Email: jiaoyong@csu.edu.cn

ABSTRACT

We propose a novel approach in noncommutative probability, which can be regarded as an analogue of good-λ inequalities from the classical case due to Burkholder and Gundy (Acta Math. **124**: 249-304, 1970). This resolves a longstanding open problem in noncommutative realm. Using this technique, we offer a new, simpler and unified approach to fundamental results in the noncommutative martingale theory, obtained earlier by Junge, Pisier, Randrianantoanina and Xu. We also present some fully new applications of good-λ approach to noncommutative probability and noncommutative harmonic analysis, including new estimates for noncommutative martingales with tangent difference sequences and sums of tangent positive operators, as well as inequalities for differentially subordinate operators which have roots in the L^p-bound for the directional Hilbert transforms on free group von Neumann algebras and the L^p-estimate for the j-th Riesz transform on group von Neumann algebras. We emphasize that all the constants obtained in this paper are of optimal orders.

Sub-Bergman Hilbert Spaces in the Unit Disk

Kehe Zhu

Department of Mathematics and Statistics State University of New York at Albany Albany, NY 12222, USA
Email: kzhu@math.albany.edu

ABSTRACT

Let H_1^∞ denote the closed unit ball of the algebra of bounded analytic functions on the unit disk $\mathbb{D}$. For $\varphi \in H_1^\infty$ consider the defect operators D_φ and $D_{\overline{\varphi}}$ for the Toeplitz operators T_φ and T_φ^*, respectively, on the weighted Bergman space A_α^2. The ranges of D_φ and $D_{\overline{\varphi}}$, denoted by $H(\varphi)$ and $H(\overline{\varphi})$ and equipped with appropriate inner products, are called sub-Bergman spaces. I will talk about the relatively new theory of sub-Bergman (and sub-Hardy) Hilbert spaces, including their reproducing kernels, the compactness of the defect operators, and the identification of $H(\varphi)$ and $H(\overline{\varphi})$ with more familiar function spaces. For example, when $\alpha > -1$, we have $H(\varphi) = H(\overline{\varphi}) = A_{\alpha-1}^2$ if and only if ϕ is a finite Blaschke product. Part of the talk is based on recent joint work with Shuaibing Luo of Hunan University.

Contributed Talks

The 7th Seminar on Functional Analysis and its Applications
1-2 March 2023
Imam Khomeini International University
Qazvin, Iran

Oral Presentation **∗ : Speaker**

ON COMPOSITION-DIFFERENTIATION OPERATORS IN BERGMAN SPACES

A. ABKAR, Y. BAYAT, S. ESKANDARI*

Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin 34194, Iran
abkar@sci.ikiu.ac.ir; y.bayat@edu.ikiu.ac.ir; S.Eskandari@edu.ikiu.ac.ir

Abstract. In this paper we aim to study conditions under which every weighted composition-differentiation operator on the Bergman space is compact.

1. Introduction

We begin by introducing some well-known functional Hilbert spaces of analytic functions in the unit disk. Let f be an analytic function in the unit disk $\mathbb{D}$. The function f is said to belong to the Hardy space H^2 if

$$\|f\|^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \mathrm{d}\theta < \infty.$$

It is easy to see that, for an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the norm of f in H^2 satisfies

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

2020 *Mathematics Subject Classification. Primary 47B33; 46E22*
Key words and phrases. Weighted Bergman space; composition-differentiation operator; compact operator.
* Speaker.

Another functional Hilbert space on the unit disk is the weighted Bergman space A^2_α consisting of all analytic functions f in the unit disk for which the integral

$$\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\alpha \mathrm{d}A(z)$$

is finite. Here α is a real parameter larger than -1, and $\mathrm{d}A(z) = \pi^{-1}\mathrm{d}x\mathrm{d}y$ is the normalized area measure in the unit disk. The norm of f is defined by

$$\|f\|^2_{A^2_\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\alpha \mathrm{d}A(z).$$

A computation reveals that for $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have

$$\|f\|^2_{A^2_\alpha} = \sum_{n=0}^{\infty} \frac{n!\,\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)}|a_n|^2,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Let $\mathcal{H}$ denote a particular functional Hilbert space of analytic functions on the open unit disk. For an analytic self-mapping φ on the unit disk, the *composition operator* $C_\varphi : \mathcal{H} \to \mathcal{H}$ is defined by

$$C_\varphi(f) = f \circ \varphi.$$

It is well-known [2, Corollary 3.7] that the composition operator is bounded on the Hardy space H^2 and on the Bergman space. Another operator which is closely related to the composition operator is the so-called differentiation operator $D(f) = f'$ provided that f' belongs to $\mathcal{H}$ as well. In the context of analytic functions, it is easy to verify that the differentiation operator is not bounded on the Hardy space H^2; since $\{z^n\}_{n\geq 1}$ is a sequence of unit vectors in the Hardy space satisfying $\|D(z^n)\| = n$. Nevertheless, for many analytic self-mappings φ on the unit disk, the operator $D_\varphi : H^2 \to H^2$ defined by

$$D_\varphi(f) = f' \circ \varphi$$

is bounded. Following [3], we call D_φ a *composition-differentiation operator*. This operator was already studied by several authors, among them S. Ohno characterized its boundedness and compactness in terms of Carleson measures; see also [1], where the authors discussed the conditions to ensure that the composition-differentiation operator is Hilbert-Schmidt.

In the year 2006 Shûichi Ohno proved that if

$$\|\varphi\|_\infty = \sup\{|\varphi(z)| : z \in \mathbb{D}\} < 1,$$

then D_φ is a Hilbert-Schmidt operator, and hence bounded and compact; see [4, Theorem 3.3]. According to [4], for a univalent self-map φ of the unit disk, the operator D_φ on the Hardy space H^2 is bounded if and only if

$$\sup_{w\in\mathbb{D}} \frac{1-|w|}{(1-|\varphi(w)|)^3} < \infty.$$

Moreover, D_φ on H^2 is compact if and only if

$$\lim_{|w|\to 1^-} \frac{1-|w|}{(1-|\varphi(w)|)^3} = 0.$$

Assuming that the symbol function φ is an analytic self map of the unit disk, we study the operator D_φ on the weighted Bergman space and provide an equivalent condition for the compactness of D_φ when $\alpha \geq 1$.

2. The Main Result

In this section we aim to characterize the compactness of composition-differentiation operator D_φ on the weighted Bergman space A^2_α. For this reason, we begin by computing the adjoint of the composition-differentiation operator $D_{\psi,\varphi}$. Recall that the Hardy, and the Bergman space are reproducing kernel Hilbert spaces. It is well-known that the reproducing kernel for the weighted Bergman space A^2_α is

$$K^\alpha_w(z) = \frac{1}{(1-\overline{w}z)^{\alpha+2}}, \quad (z,w) \in \mathbb{D}\times\mathbb{D}.$$

This means that for each $f \in A^2_\alpha$ we have

$$f(w) = \langle f, K^\alpha_w\rangle = \int_{\mathbb{D}} \frac{f(z)}{(1-\overline{z}w)^{\alpha+2}} \mathrm{d}A_\alpha(z),$$

where

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \mathrm{d}A(z).$$

It now follows that

$$f'(w) = \int_{\mathbb{D}} f(z) \frac{(\alpha+2)\overline{z}}{(1-\overline{z}w)^{2\alpha+3}} \mathrm{d}A_\alpha(z).$$

Now, the uniqueness of the kernel function implies that

$$K^{(1)}_{\alpha,w}(z) := \frac{(\alpha+2)z}{(1-\overline{w}z)^{2\alpha+3}}, \quad (z,w) \in \mathbb{D}\times\mathbb{D}$$

is the reproducing kernel corresponding to the functional $f \mapsto f'(w)$ defined on the weighted Bergman space A^2_α. In other words, for each $f \in A^2_\alpha$ we have

$$f'(w) = \langle f, K^{(1)}_{\alpha,w}\rangle, \quad w \in \mathbb{D}.$$

Lemma 2.1. *Let φ be an analytic self map on $\mathbb{D}$, and let $\psi : \mathbb{D} \to \mathbb{C}$ be an analytic function such that $D_{\psi,\varphi}$ is bounded on A^2_α. Then $D^*_{\psi,\varphi}(K^\alpha_w) = \overline{\psi(w)} K^{(1)}_{\alpha,\varphi(w)}$.*

Theorem 2.2. *Let φ be an analytic self map of the unit disk, and $\alpha \geq 1$. Then the operator $D_\varphi : A^2_\alpha(\mathbb{D}) \to A^2_\alpha(\mathbb{D})$ is compact if and only if*

$$\lim_{|w|\to 1^-} \left(\frac{1-|w|^2}{(1-|\varphi(w)|^2)^2} \right)^{\alpha+2} = 0. \tag{2.1}$$

Proof. Let D_φ be compact and

$$k_w^\alpha(z) = \sqrt{(1-|w|^2)^{\alpha+2}}K_w^\alpha(z) = \frac{\sqrt{(1-|w|^2)^{\alpha+2}}}{(1-\overline{w}z)^{\alpha+2}}$$

be the normalized reproducing kernels of $A^2_\alpha(\mathbb{D})$. Let

$$K^{(1)}_{\alpha,w}(z) = \frac{(\alpha+2)z}{(1-\overline{w}z)^{2\alpha+3}}$$

be the reproducing kernel corresponding to the functional $f \mapsto f'(w)$ on the weighted Bergman space $A^2_\alpha(\mathbb{D})$. By Lemma 2.1, $D^*_\varphi(K_w^\alpha) = K^{(1)}_{\alpha,\varphi(w)}$. Therefore,

$$\begin{aligned}\|D^*_\varphi(k_w^\alpha)\|^2 &= (1-|w|^2)^{\alpha+2}\|K^{(1)}_{\alpha,\varphi(w)}\|^2\\ &= (1-|w|^2)^{\alpha+2}\frac{(\alpha+2)\left[1+(\alpha+2)|\varphi(w)|^2\right]}{(1-|\varphi(w)|^2)^{2\alpha+4}}\\ &= (\alpha+2)\left[1+(\alpha+2)|\varphi(w)|^2\right]\left(\frac{1-|w|^2}{(1-|\varphi(w)|^2)^2}\right)^{\alpha+2}.\end{aligned}$$

But $k_w^\alpha \to 0$ weakly as $|w| \to 1^-$ (see [2, Theorem 2.17]), so that the compactness of D^*_φ implies that $\|D^*_\varphi k_w^\alpha\| \to 0$ as $|w| \to 1^-$. This yields

$$\lim_{|w|\to 1^-}\left(\frac{1-|w|^2}{(1-|\varphi(w)|^2)^2}\right)^{\alpha+2} = 0.$$

Conversely, assume that φ satisfies the condition (2.1). Let (f_n) be a bounded sequence in $A^2_\alpha(\mathbb{D})$ that converges weakly to zero. We proceed to show that $\|D_\varphi f_n\| \to 0$ as $n \to \infty$. $\square$

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[1] A. Babaei, A. Abkar, *Weighted composition-differentiation operators on the Hardy and Bergman spaces*, Bull. Iran. Math. Soc. **48**(2022), 3637–3658

[2] C.C. Cowen, B.D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, 1995

[3] M. Fatehi, C.N.B. Hammond, *Composition-differentiation operators on the Hardy space*, Proc. Am. Math. Soc. **148** (2020), no. 7, 2893–2900

[4] S. Ohno, *Products of composition and differentiation between Hardy spaces*, Bull. Austr. Math. Soc. **73** (2006), no. 2, 235–243

Oral Presentation

ON DIFFERENTIAL SUBORDINATION FOR ANALYTIC FUNCTIONS WITH FIXED SECOND COEFFICIENT

MOSTAFA AMANI

Faculty of Science, Department of Mathematics, Urmia University, Urmia, Iran
mos.amani@urmia.ac.ir , mos_amani@yahoo.com

Abstract. Some new results closely related to the generalized Briot-Bouquet differential subordination are investigatedin in a new approach for functions with fixed second coefficient.

1. Introduction and preliminaries

Let $\mathcal{H}$ be the class of analytic functions in the unit disc $\mathbb{U} = \{z : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote by $\mathcal{H}_\beta[a, b]$ and $\mathcal{A}_{n,b}$ sets of analytic functions with fixed initial coefficient, respectively, as below:

$$\mathcal{H}_\beta[a, n] = \left\{p \in \mathcal{H} : p(z) = a + \beta z^n + a_{n+1}z^{n+1} + \cdots \right\},$$

and

$$\mathcal{A}_{n,b} = \{f \in \mathcal{H} : f(z) = z + bz^{n+1} + \cdots , z \in \mathbb{U}\}.$$

where β and $b \in \mathbb{C}$ are fixed. Here, we assume that β and b are positive real numbers. The concept of subordination was introduced to describe a relation between pairs of analytic functions; Let $f(z)$ and $g(z)$ be members of the class $\mathcal{H}$. we sayvthat $f(z)$ is subordinate to $g(z)$ and write by $f(z) \prec g(z)$ if there exists a function $w(z) \in \mathcal{H}$ with $w(0) = 0, |w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. It is easy to see that when $g(z)$ is univalent in $\mathbb{U}$, then $f(0) = g(0)$ *and* $f(\mathbb{U}) \subseteq g(\mathbb{U})$ is the equivalent definition of subordination. The start of differential subordination theory began in 1974 by

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Miller, Mocanu and Reade [7]. Then in 1981, Miller and Mocanu [6] introduced the analogues differential subordination and built the theory for this type of differential implications. In 2011, Rosihan, Nagpal and Ravichandran [12] extended the theory of second-order differential subordination for functions with fixed initial coefficient. This led to many results related to the differential subordination being extended and improved, that recently have published several articles on the application of this new result(For example, see [1, 2, 3, 4]). In this paper, by extension of the Nunokawa lemma [9, 10] due to author et al. [2], some new results closely related to the generalized Briot-Bouquet differential subordination are investigated in a new approach for functions with fixed second coefficient. First, we need some of the following fundamental definition and theorems.

Definition 1.1. ([8], [p.24]) Assume that $\mathbf{Q}$ is the set of functions q that are analytic and injective on $\overline{\mathbb{U}}\backslash E(q)$ with $E(q) := \{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\}$, and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(q)$.

Lemma 1.2. [12] *Let $q \in \mathbf{Q}$ with $q(0) = a$, and $p \in \mathcal{H}_\beta[a, n]$ with $p(z) \not\equiv a$. If $p \not\prec q$, then there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash E(q)$ for which $p(z_0) = q(\zeta_0)$, $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{U})$ and $z_0p'(z_0) = m\zeta_0q'(\zeta_0)$. Moreover*

$$\mathfrak{Re}\left\{1 + \frac{z_0p''(z_0)}{p'(z_0)}\right\} \geq m\ \mathfrak{Re}\left\{1 + \frac{\zeta_0q''(\zeta_0)}{q'(\zeta_0)}\right\}, \tag{1.1}$$

for some $m \geq n + (|q'(0)| - \beta|z_0|^n)/(|q'(0)| + \beta|z_0|^n)$

Lemma 1.3. [1] *Let $p \in \mathcal{H}_\beta[1, n]$ and $p(z) \neq 0$ in $\mathbb{U}$. If there exist $z_0 \in \mathbb{U}$ such that $|\mathfrak{arg}\ p(z)| < \pi\alpha/2$ for $|z| < |z_0|$ and $|\mathfrak{arg}\ p(z_0)| = \pi\alpha/2$ where $\alpha > 0$. Then we have*

$$\frac{z_0p'(z_0)}{p(z_0)} = i\alpha m\ , \qquad m \leq -\frac{1}{2}\left(a + \frac{1}{a}\right)\left(n + \frac{2\alpha - \beta}{2\alpha + \beta}\right)$$

when $\mathfrak{arg}\ p(z_0) = -\frac{\pi\alpha}{2}$ and

$$\frac{z_0p'(z_0)}{p(z_0)} = i\alpha m\ , \qquad m \geq \frac{1}{2}\left(a + \frac{1}{a}\right)\left(n + \frac{2\alpha - \beta}{2\alpha + \beta}\right)$$

when $\mathfrak{arg}\ p(z_0) = \frac{\pi\alpha}{2}$ where $p(z_0)^{\frac{1}{\alpha}} = \pm ia$ and $0 \leq \beta \leq 2\alpha$.

2. Main Results

Theorem 2.1. *Let $B(z)$ and $C(z)$ be analytic in $\mathbb{U}$ with $|\mathfrak{Im}\{C(z)\}| < \mathfrak{Re}\{B(z)\}$. If $p(z) \in \mathcal{H}_\beta[1, n]$, $0 \leq \beta \leq 2$, and if*

$$|\mathfrak{arg}\{B(z)zp'(z) + C(z)p(z)\}| < \frac{\pi}{2} + t(z), \tag{2.1}$$

where

$$t(z)=\begin{cases}\mathfrak{arg}\left\{B(z)i\left[\frac{2(n+1)+\beta(n-1)}{2+\beta}\right]+C(z)\right\}:=\tau & when \quad \tau\in[0,\pi/2],\\ \\ \mathfrak{arg}\left\{B(z)i\left[\frac{2(n+1)+\beta(n-1)}{2+\beta}\right]+C(z)\right\}-\frac{\pi}{2}:=\tau' & when \quad \tau\in(\pi/2,\pi],\end{cases} \tag{2.2}$$

then $\mathfrak{Re}\{p(z)\}>0. \quad (z\in\mathbb{U})$

Remark 2.2. Theorem 2.1 improves a result due to Miller and Mocanu [See [5], p. 208]. Also, it extends a result due to Nunokawa et al. [See [11], p. 3].

Corollary 2.3. *Let* $g(z)\in\mathcal{H}_\beta[1,n],\ 0\le\beta\le 2$ *with*

$$|\mathfrak{Im}\left\{\frac{zg'(z)}{g(z)}\right\}|<1,$$

and let $f\in\mathcal{A}_{n,b}$. *Suppose that*

$$|\mathfrak{Im}\{g(z)f'(z)\}|<\frac{\pi}{2}+\nu(z), \quad (z\in\mathbb{U})$$

where

$$\nu(z)=\begin{cases}\mathfrak{arg}\left\{i\left[\frac{2(n+1)+(\beta+b)(n-1)}{2+\beta+b}\right]+1-\frac{zg'(z)}{g(z)}\right\}:=\lambda & when \quad \lambda\in[0,\pi/2],\\ \\ \mathfrak{arg}\left\{i\left[\frac{2(n+1)+(\beta+b)(n-1)}{2+\beta+b}\right]+1-\frac{zg'(z)}{g(z)}\right\}-\frac{\pi}{2}:=\lambda' & when \quad \lambda\in(\pi/2,\pi].\end{cases}$$

Then we have

$$\mathfrak{Re}\left\{\frac{g(z)f(z)}{z}\right\}>0. \quad (z\in\mathbb{U})$$

Remark 2.4. By taking $\beta+b=2$ and $n=1$, Corollary 2.3 reduces to a result obtained by Nunokawa et al. [See [11], p. 5]. Also, it improves a result due to Miller and Mocanu [See [5], p. 208]

Theorem 2.5. *Let* $B(z)$ *and* $C(z)$ *be analytic in* $\mathbb{U}$ *with* $B(z)\neq 0$. *Suppose that*

$$\mathfrak{Re}\left\{\frac{C(z)}{B(z)}\right\}\ge -T(n,\beta), \quad (z\in\mathbb{U}) \tag{2.3}$$

where

$$T(n,\beta)=\frac{n+1+\beta(n-1)}{1+\beta}, \tag{2.4}$$

for $0\le\beta\le 1$ *and* $n\ge 1$. *If* $p(z)\in\mathcal{H}_\beta[0,n]$, *and if*

$$|B(z)zp'(z)+C(z)p(z)|<|B(z)+C(z)|, \quad (z\in\mathbb{U}) \tag{2.5}$$

then $|p(z)|<1. \quad (z\in\mathbb{U})$

Theorem 2.6. *Let* $B(z)$ *and* $C(z)$ *be analytic in* $\mathbb{U}$ *with* $B(z)\neq 0$. *Suppose that*

$$\mathfrak{Im}\left\{\frac{C(z)}{B(z)}\right\}\ge\frac{T(n,\beta)}{|B(z)|}, \quad (z\in\mathbb{U}) \tag{2.6}$$

where $T(n,\beta)$ *is the same as* (2.4) *with* $0 \le \beta \le 1$ *and* $n \ge 1$. *If* $p(z) \in \mathcal{H}_\beta[0,n]$, *and if*

$$|B(z)zp'(z) + C(z)p(z)| < \sqrt{1 + |B(z)|^2 \left[\frac{zp'(z)}{p(z)} + \mathfrak{Re}\left\{\frac{C(z)}{B(z)}\right\}\right]^2}, \quad (2.7)$$

in $\mathbb{U}$, *then* $|p(z)| < 1$. $(z \in \mathbb{U})$.

Remark 2.7. Theorem 2.6 improves a result due to Miller and Mocanu [See [5], p .207].

Theorem 2.8. *If* $p \in \mathcal{H}_\beta[0,n]$ *with* $0 \le \beta \le 1$ *and* $n \ge 1$, *then*

$$|zp'(z)| + \left|\frac{z^2p''(z)}{p(z)}\right| < \left[n + \frac{1-\beta}{1+\beta}\right]^2 \quad (2.8)$$

implies that $|p(z)| < 1$.

Remark 2.9. By taking $\beta = n = 1$, Theorem 2.8 reduces to a result due to Miller and Mocanu [See [5], p .207].

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Oral Presentation

HERMITE-HADAMARD INTEGRAL INEQUALITY FOR SOME TYPES OF CONVEX FUNCTIONS

MEHDI ASADI

Department of Mathematics, Zanjan Branch, Islamic Azad University, Zanjan, Iran
masadi.azu@gmail.com; me.asadi@iau.ac.ir

Abstract. In this paper, we verify Hermite-Hadamard integral inequality on some types of convex functions. Previous results are some part of our consequences.

1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I and $x, y \in I$. Then (trapezium inequality)

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This double inequality is known in the literature as the Hermite-Hadamard (HH) integral inequality for convex functions.

2. Preliminary

Definition 2.1 ([12]). Let $m, t, \alpha \in [0,1]$. Then the real number set $C \subseteq \mathbb{R}$ is said to be

(1) convex if $tx + (1-t)y \in C$;
(2) m-convex if $tx + (1-t)my \in C$;
(3) (α, m)-convex if $t^{\alpha}x + (1-t^{\alpha})my \in C$;

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for all $x, y \in C$ and $t, m \in [0, 1]$.

Definition 2.2 ([3, 4, 9, 10, 12])**.** Let $m \in [0, 1]$ and $C \subseteq \mathbb{R}$. A function $f : C \subset \mathbb{R} \to \mathbb{R}$ is said to be an

(1) convex, if C be a convex set and

$$f(tx + (1-t)y] \leq tf(x) + (1-t)f(y);$$

(2) m-convex, if C be a m-convex set and

$$f(tx + (1-t)my] \leq tf(x) + (1-t)mf(y);$$

(3) (α, m)-convex, if C be a (α, m)-convex set and

$$f(tx + (1-t)my] \leq t^{\alpha}f(x) + (1-t^{\alpha})mf(y);$$

(4) f is concave if $-f$ is convex;
(5) f is m-concave if $-f$ is m-convex.
(6) $f : [a, b] \to \mathbb{R}$ is star shaped if $f(tx) \leq tf(x)$ for all $t \in [0, 1]$ and $x \in [a, b]$.

for all $x, y \in C$ and $t, m \in [0, 1]$.

Remark 2.3. ([10, 7])

(1) When $t = 1$, we get $f(my) \leq mf(y)$ for all $x, y \in I$, means the function f is sub-homogeneous.
(2) If f was convex function and $m = 1$, it would be m-convex function.

Lemma 2.4. ([4, 7])

(1) *If $f : C \to \mathbb{R}$ is m-convex and $0 \leq n < m \leq 1$, then f is n-convex.*
(2) *Let $f, g : [a, b] \to \mathbb{R}$, $a \geq 0$. If f is n-convex and g is m-convex, with $n \leq m$, then $f + g$ and αf, $\alpha \geq 0$ a constant, are n-convex.*
(3) *Let $f : [0, a] \to \mathbb{R}$, $g : [0, b] \to \mathbb{R}$, with* $\mathrm{renge}(f) \subseteq [0, b]$. *If f and g are m-convex and g is increasing, then $g \circ f$ is m-convex on $[0, a]$.*
(4) *If $f, g : [0, a] \to \mathbb{R}$ are both nonnegative, increasing and m-convex, then fg is m-convex.*

3. Main results

Put $\mathrm{co}(A) = \{f : f \text{ is convex}\}$ and $\mathrm{co}_m(B) = \{f : f \text{ is } m-\text{convex}\}$. So $\mathrm{co}_m(B) \subsetneqq \mathrm{co}(A)$, See more detail in [1].

Theorem 3.1. *Let $m \in [0, 1]$ and $C \subseteq \mathbb{R}$ and function $f : C \subset \mathbb{R} \to \mathbb{R}$ be a m-convex function on an interval C and $a, b \in C$. If $a + mb = r + s$ for every r and s. Then*

$$\begin{aligned} f\left(\frac{a+mb}{2}\right) &\leq \frac{1}{2}\left(\frac{1}{r-a}\int_a^r f(u)du + m\frac{1}{mb-s}\int_s^{mb} f(u)du\right) \\ &\leq \frac{f(r)+mf(s)}{2} + \frac{f(a)+mf(mb)}{2}. \end{aligned}$$

Corollary 3.2. *By hypothesis of Threorem 3.1 we have:*

$$\frac{1}{2}\left(\frac{1}{r-a}\int_a^r f(u)du + m\frac{1}{mb-s}\int_s^{mb} f(u)du\right) \leq \frac{f(r)+f(a)}{2} + m\frac{f(s)+mf(b)}{2}.$$

Corollary 3.3.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{4} + \frac{1}{2}f\left(\frac{a+b}{2}\right).$$

An other version of Theorem 3.1:

Theorem 3.4. *Let $m \in [0,1]$ and $C \subseteq \mathbb{R}$ and function $f : C \subset \mathbb{R} \to \mathbb{R}$ be a m-convex function on an interval C and $a, b \in C$. Then*

$$\begin{aligned} f\left(\frac{a+mb}{2}\right) &\leq \frac{1}{mb-a}\left(\int_a^{\frac{mb+a}{2}} f(u)du + m\int_{\frac{mb+a}{2}}^{mb} f(u)du\right) \\ &\leq \frac{f(a)+mf(b)}{2}\left(\frac{m+1}{4}\right) + \frac{f(a)+m^2f(b)}{4} \\ &= \frac{(3m+1)(mf(b))+(m+3)f(a)}{8}. \end{aligned}$$

Corollary 3.5. *By hypothesis of Theorem 3.4:*

$$\frac{1}{mb-a}\left(\int_a^{\frac{mb+a}{2}} f(u)du + m\int_{\frac{mb+a}{2}}^{mb} f(u)du\right) \leq \frac{(3m+1)(mf(b))+(m+3)f(a)}{8}.$$

If we put $m = 1$ in Theorem 3.4, then we will find Hermite-–Hadamard (HH) integral inequality.

For more details and some of related references see [2, 5, 6, 8, 11, 13].

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TOPOLOGICALLY TRANSITIVITY ON NON-SEPARABLE BANACH SPACES

MEYSAM ASADIPOUR*, ROHOLLAH PARVINIANZADEH

Department of Mathematics, College of Sciences, University of Yasouj, Yasouj, 75918-74934, Iran
Meysam.asadipour@yahoo.com
r.parvinian@yu.ac.ir

Abstract. When X is an infinite-dimensional Banach space and $B(X)$ denotes the Banach algebra of all bounded linear operators on X, then we will consider topological transitive operators in $B(X)$. In this paper, we will investigate the density of the range of a topological transitive operator on X.

1. Introduction and preliminaries

Assume that X is a Banach space and $B(X)$ denotes the Banach algebra of all bounded linear operators on X. Consider an operator $T \in B(X)$. If for every pair U, V of nonempty open subsets of X, there is a positive integer n so that subset $T^n(U) \cap V$ is nonempty, then the operator T is topological transitive. The first example of topological transitive operators was presented by Birkhoff in [4]. If the underlying space is considered as a separable Banach space, then it is a simple matter to see that topological transitivity is equivalent to hypercyclicity. To be more clear, if B is a subset of X, then the orbit of B under T is the set $orb(T, B) = \{T^n x;\ x \in B, n = 0, 1, 2, \cdots\}$. if B is a singleton $\{x\}$ and $\overline{orb(T, B)} = X$, then T is called a hypercyclic operator and x is a hypercyclic vector for T. If $B = \{\lambda x;\ \lambda \in \mathbb{C}\}$ for some

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* Speaker.

vector $x \in X$ and $orb(T, B)$ is a dense subset of X, then this operator is said to be a supercyclic operator and the vector x is a supercyclic vector for T. Note that, since $\{T^n x;\ x \in X, n = 0, 1, 2, \cdots\} \subseteq T(X)$, so a topological transitive operator T on a separable Banach space has dense range. We recall that, there is no hypercyclic operator on a finite-dimensional Banach space. On the other side, Ansari [1] showed that every infinite-dimensional separable Banach space admits a topological transitive operator. Hence, in the following X is an infinite-dimensional Banach space.

It is worth pointing out that when X is an infinite-dimensional non-separable Banach space, then obviously X cannot support hypercyclic operators. However, it is well known that topologically transitive operators may exist in X, see for instance [5].

Now it is natural to raise the following question.

Problem. Let T be a topological transitive operator T on a non-separable Banach space X. Dose it have dense range?.

In the next section we will give positive answer to this question as the main result of this paper. For details and references on topological transitive operators on non-separable and separable Banach spaces see [2] and [3].

2. Topological transitivity and whose equivalent assertions

The second section deals with some assertions which are equivalent to topological transitivity. We emphasize that in this section the underlying space X is an arbitrary Banach space, so it may be a non-separable Banach space. Then we will give two different proofs of the density of the range of a topological transitive operator on X.

Theorem 2.1. *Let T be an operator on a Banach space X. Then the following are equivalent.*
i) T is topological transitive,
ii) $\overline{\bigcup_{n=0}^{\infty} T^n(U)} = X$, whenever U is an arbitrary open subset of X,
iii) $\overline{\bigcup_{n=0}^{\infty} T^{-n}(U)} = X$, whenever U is an arbitrary open subset of X,
iv) every proper open T^{-1}-invariant subset of X is dense in X,
v) every proper closed T-invariant subset of X is nowhere dense in X.

Proof. Since $T^n(U) \cap V \neq \emptyset$ and $T^{-n}(V) \cap U \neq \emptyset$ are equivalent, so it is evident that the assertions *(i)*, *(ii)* and *(iii)* are equivalent. Thus we only need to prove *(i)* $\Longleftrightarrow$ *(iv)*, because *(i)* $\Longleftrightarrow$ *(v)* can be proved in much the same way as *(i)* $\Longleftrightarrow$ *(iv)*. For this goal, in the first step assume that U is an open subset of X and $T^{-1}(U) \subseteq U$. To obtain a contradiction, suppose that there exists an $x \in X \setminus \overline{U}$. Now if we cosider a neighbourhood V_x of x, then *(i)* implies that $T^n(V_x) \cap U \neq \emptyset$, for some $n \in \mathbb{N}$. Consequently, the contradiction

$$\emptyset \neq T^{-n}(U) \cap V_x \subseteq T^{-n+1}(U) \cap V_x \subseteq \cdots U \cap V_x = \emptyset,$$

shows that *(i)* implies *(iv)*.
Conversely, again to obtain a contradiction suppose that

$$\left(\bigcup_{n=0}^{\infty} T^{-n}(U)\right) \cap V = \emptyset,$$

for some non-empty open subsets U, V of X. This means that the open subset $\widehat{U} := \bigcup_{n=0}^{\infty} T^{-n}(U)$ is not dense in X. The assertion *(iv)* implies that $T^{-1}(\widehat{U}) \nsubseteq \widehat{U}$ which is a contradiction because it is easy to check that $T^{-1}(\widehat{U}) \subseteq \widehat{U}$. Therefore the proof of *(iv)* $\Rightarrow$ *(i)* is completed.

□

We can get the following corollary from the assertion *(iii)* in the previous theorem.

Corollary 2.2. *Every topological transitive operator on a separable or non-separable has dense range.*

Proof. Let V be an arbitrary open subset of X. the assertion *(iii)* implies that $T^{-m}(V) \cap V \neq \emptyset$, for some $m \in \mathbb{N}$, so there exists a vector $x \in V$ such that $T^m x \in V$. This means that $T^m x \in T(X)$ and consequently $V \cap T(X)$ is non-empty. Therefore $T(X)$ is dense in X. □

It is interesting to know that the above result can be derived from *(v)*. Thus we give a different proof of the above corollary.

Proof. Assume that T is topological transitive operator and also assume that λ is an eigenvalue of T^*. If x^* is a corresponding eigenvector to λ, then one of the subsets $\{x : |x^*(x)| \geq 1\}$ or $\{x : |x^*(x)| \leq 1\}$ is an invariant under T with non-empty interior. This contrary to the assertion *(v)* and consequently $\sigma_p(T^*) = \emptyset$. Since $\sigma_p(T^*) = \emptyset$ is equivalent to the density of the range of T, so the proof is completed. □

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Oral Presentation **$*$: Speaker**

COMPLETE CONTINUITY OF WEIGHTED COMPOSITION-DIFFERENTIATION OPERATORS IN HARDY SPACE

A. BABAEI* AND A. ABKAR

Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin 34194, Iran
a.babaei@edu.ikiu.ac.ir; abkar@sci.ikiu.ac.ir

Abstract. In this paper we explore conditions under which every weighted composition-differentiation operator on the Hardy space $H^1(\mathbb{D})$ is completely continuous.

1. Introduction

Let $\mathcal{X}$ be a Banach space of analytic functions on the unit disk, and let φ be an analytic self-mapping on the unit disk. The *composition operator* $C_\varphi : \mathcal{X} \to \mathcal{X}$ is defined by
$$C_\varphi(f) = f \circ \varphi.$$
It is well-known that the composition operator is bounded on the Hardy space H^p and on the Bergman space A^p where p is a positive number. For a function $\psi \in \mathcal{X}$, the *weighted composition operator* $C_{\psi,\varphi} : \mathcal{X} \to \mathcal{X}$ is defined by
$$C_{\psi,\varphi}(f) = \psi \cdot f \circ \varphi.$$
Similarly, we can define the *composition-differentiation operator* $D_\varphi : \mathcal{X} \to \mathcal{X}$ by
$$D_\varphi(f) = f' \circ \varphi.$$

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* Speaker.

In most cases the functional Banach space $\mathcal{X}$ equals either the Hardy space H^p or the Bergman space A^p. According to [4, Corollary 3.2], for a univalent self-map φ of the unit disk, the operator D_φ on the Hardy space H^2 is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{1-|z|}{(1-|\varphi(z)|)^3} < \infty.$$

Moreover, D_φ is compact on H^2 if and only if

$$\lim_{|z|\to 1} \frac{1-|z|}{(1-|\varphi(z)|)^3} = 0.$$

Now, let ψ be an analytic function on the unit disk, and define the *weighted composition-differentiation* operator $D_{\psi,\varphi} : \mathcal{X} \to \mathcal{X}$ by the following relation:

$$D_{\psi,\varphi}(f) = \psi \cdot f' \circ \varphi.$$

This operator was recently studied in [1] and [3].

An operator $T : \mathcal{X} \to \mathcal{X}$ is said to be *completely continuous* if $x_n \to x$ weakly in $\mathcal{X}$, implies $\|Tx_n - Tx\| \to 0$. It is well-known that on a Banach space $\mathcal{X}$, every compact operator is completely continuous. On the other hand, if the Banach space $\mathcal{X}$ is reflexive, then completely continuous operators are compact. In this paper we shall focus on the non-reflexive Hardy space H^1, and try to find conditions under which the weighted composition-differentiation operator $D_{\psi,\varphi}$ is completely continuous. We shall provide characterizations for the complete continuity of this operator in terms of ψ and φ. More precisely, we prove that $D_{\psi,\varphi}$ is completely continuous if and only if $\psi = 0$ almost everywhere in $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$.

2. Preliminaries

An analytic function f on the unit disk is said to belong to the Hardy space $H^p = H^p(\mathbb{D})$ if

$$\|f\|_{H^p}^p = \sup_{0\le r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

For $1 \le p < \infty$, the Hardy space H^p is a Banach space of analytic functions, and for $p = 2$ it is a Hilbert space with the following inner product:

$$\langle f, g\rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta})\overline{g^*(e^{i\theta})} d\theta,$$

where

$$f^*(e^{i\theta}) := \lim_{r\to 1^-} f(re^{i\theta})$$

is the boundary function of f. It is easy to see that for $f \in H^2$ with Taylor series $f(z) = \sum_{n=0}^\infty a_n z^n$, the norm of f is given by

$$\|f\|_{H^2}^2 = \sum_{n=0}^\infty |a_n|^2.$$

Recall that an operator $T : \mathcal{X} \to \mathcal{X}$ is compact if for every bounded sequence (x_n) in $\mathcal{X}$, the sequence (Tx_n) has a convergent subsequence. We remark that for $1 < p < \infty$, the Hardy space H^p is reflexive; meaning that it is isometrically isomorphic with its dual. We know that on reflexive Banach spaces, an operator T is compact if and only if it is completely continuous. In this paper, we concentrate on the non-reflexive Banach space H^1 and the composition-differentiation operator $D_{\psi,\varphi}$ on H^1. We will find conditions on the function φ to ensure that the operator $D_{\psi,\varphi}$ is completely continuous on H^1.

3. Main Result

In the following theorem we shall characterize the complete continuity of composition-differentiation operator in terms of ψ and φ.

Theorem 3.1. *Let $\psi \in H^1$ and φ be a self-map on $\mathbb{D}$. Assume that $D_{\psi,\varphi}$ is bounded on H^1. Then $D_{\psi,\varphi}$ is completely continuous on H^1 if and only if $\psi = 0$ almost everywhere in $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$.*

Proof. Let $D_{\psi,\varphi}$ be completely continuous, and let $\mathbb{T}$ denote the unit circle. Assume that $f \in L^\infty(\mathbb{T})$ and let $\hat{f}(n)$ be its n-th Fourier coefficient. By Riemann-Lebesgue lemma we have

$$\int_{\mathbb{T}} f(z)\overline{z}^n \mathrm{d}m = \hat{f}(n) \to 0, \quad n \to \infty.$$

This means that $\{z^n\}$ converges to zero weakly in $L^1(\mathbb{T})$, and hence weakly in H^1. Since $D_{\psi,\varphi}$ is completely continuous, it follows that

$$\|D_{\psi,\varphi}(z^n)\|_{H^1} \to 0, \quad n \to \infty.$$

On the other hand, for each $n \in \mathbb{N}$,

$$\begin{aligned} 0 \le \int_{\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}} |\psi|\mathrm{d}m &\le \int_{\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}} n|\psi|\mathrm{d}m \\ &= \int_{\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}} n|\psi||\varphi|^{n-1}\mathrm{d}m \\ &\le \int_{\mathbb{T}} n|\psi||\varphi|^{n-1}\mathrm{d}m \\ &= \|D_{\psi,\varphi}(z^n)\|_{H^1} \to 0, \quad n \to \infty. \end{aligned}$$

Therefore the integral on the left-hand side must be zero, from which it follows that $\psi = 0$ almost everywhere in $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$.

Conversely, Let (f_n) be a weak null sequence in H^1. It follows that $f'_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Using this fact together with the assumption that $\psi = 0$ almost everywhere in $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$, we conclude that

$$D_{\psi,\varphi}(f_n)(e^{i\theta}) = \psi(e^{i\theta})f'_n(\varphi(e^{i\theta})) \to 0, \quad a.e.\,\mathrm{in}\,\mathbb{T}.$$

It now follows that $D_{\psi,\varphi}(f_n)$ converges to zero in measure in $L^1(\mathbb{T})$ (see [5, page 74]). Moreover, the boundedness of $D_{\psi,\varphi}$ on H^1 implies that $D_{\psi,\varphi}(f_n) \to 0$ in the weak topology of H^1, and hence in the weak topology of $L^1(\mathbb{T})$. Finally, we invoke the fact that weak convergence of a given sequence together with its convergence in measure implies its norm convergence (see [2, page 295]), that is, $\|D_{\psi,\varphi}(f_n)\|_{H^1} \to 0$ as $n \to \infty$. $\square$

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UNIFORM CONVERGENCE OF SEQUENCES OF COMPOSITION OPERATORS IN HARDY SPACE

A. BABAEI* AND A. ABKAR

Department of Pure Mathematics, Imam Khomeini International University, Qazvin 34194, Iran
a.babaei@edu.ikiu.ac.ir; abkar@sci.ikiu.ac.ir

Abstract. We study the convergence of composition operators with respect to weak operator topology as well as strong operator topology on the Hardy space of analytic functions in the unit disk.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space of analytic functions on the unit disk. For instance, $\mathcal{H}$ is the Hardy space H^2, or the Bergman space A^2. Given an analytic self-mapping φ on the unit disk, the *composition operator* $C_\varphi : \mathcal{H} \to \mathcal{H}$ is defined by

$$C_\varphi(f) = f \circ \varphi.$$

It is well-known that the composition operator is bounded on the Hardy space H^2 and

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{1/2} \leq \|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2}.$$

For a function $\psi \in \mathcal{H}$, the *weighted composition operator* $C_{\psi,\varphi} : \mathcal{H} \to \mathcal{H}$ is defined by

$$C_{\psi,\varphi}(f) = \psi \cdot f \circ \varphi.$$

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In the same manner we define the operator $D_\varphi : \mathcal{H} \to \mathcal{H}$ by

$$D_\varphi(f) = f' \circ \varphi.$$

According to [5, Corollary 3.2], for a univalent self-map φ of the unit disk, the operator D_φ on the Hardy space H^2 is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{1-|z|}{(1-|\varphi(z)|)^3} < \infty.$$

Moreover, the operator D_φ on H^2 is compact if and only if

$$\lim_{|z|\to 1} \frac{1-|z|}{(1-|\varphi(z)|)^3} = 0.$$

Now, let ψ be an analytic function on the unit disk, and define the *weighted composition-differentiation* operator $D_{\psi,\varphi} : \mathcal{H} \to \mathcal{H}$ by the following relation:

$$D_{\psi,\varphi}(f) = \psi \cdot f' \circ \varphi.$$

Our goal in this paper is to study the relationships between convergence of the sequence of operators D_{ψ_n,φ_n} in operator topologies from one hand, and the convergence of the sequences of functions ψ_n and φ_n on the other hand. G. Gunatillake's paper [3] studied the relationship between convergence of weighted composition operators C_{ψ_n,φ_n}, and the convergence of $\{\psi_n\}$ and $\{\varphi_n\}$. This result was extended by S. Mehrangiz and B. Khani-Robati [4] to generalized weighted composition operators on Bloch type spaces. Here we intend to generalize Gunatillake's result to weighted composition-differentiation operator $D_{\psi,\varphi}$ in the setting of classical Hardy spaces. More specifically, let $\mathcal{B}(H^2)$ denote the Banach algebra of all bounded linear operators on the Hilbert space H^2. It is rather well-known that the dual space of $\mathcal{B}(H^2)$ is too big, so that the weak and weak-star topology of this space is not so clear. For this reason, it is customary to equip this space with the weak operator topology, the strong operator topology, and the uniform operator topology. We intend to have a characterization of the convergence of D_{ψ_n,φ_n} to $D_{\psi,\varphi}$ with respect to operator topologies in terms of the convergence of $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in the weak and strong operator topologies of H^2.

2. Preliminaries

Let f be an analytic function in the unit disk $\mathbb{D}$. The function f is said to belong to the Hardy space H^2 if

$$\|f\|^2 = \sup_{0\le r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

It is easy to see that for an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the norm of f in H^2 is given by

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

It is well-known that for $f \in H^2$, the radial limit

$$f^*(e^{i\theta}) := \lim_{r\to 1^-} f(re^{i\theta}) = \lim_{r\to 1^-} f_r(e^{i\theta})$$

for almost every $\theta \in [0, 2\pi]$ exists. The function f^* is known as the radial function of f. The space H^2 is a functional Hilbert space, and its inner product is given by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f^*(e^{i\theta})\overline{g^*(e^{i\theta})}d\theta.$$

Since the evaluation functionals are bounded, the Hardy space is a reproducing kernel Hilbert space; this means that for each $w \in \mathbb{D}$, there is a function

$$K_w(z) = \frac{1}{1-\overline{w}z} \in H^2$$

such that every $f \in H^2$ has the following representation

$$f(w) = \langle f, K_w\rangle = \frac{1}{2\pi}\int_0^{2\pi} f^*(e^{i\theta})\overline{K_w^*(e^{i\theta})}d\theta.$$

It is also well-known that the functional $w \mapsto f'(w)$ is bounded on H^2 ([**?**, Theorem 2.16]). It then follows from the Riesz representation theorem that there is a function $K_w^{(1)} \in H^2$ such that

$$f'(w) = \langle f, K_w^{(1)}\rangle, \quad f \in H^2.$$

It turns out that (see [2])

$$K_w^{(1)}(z) = \frac{z}{(1-\overline{w}z)^2}, \quad (z, w) \in \mathbb{D}\times\mathbb{D}.$$

In [1], we have proved the following theorems on the convergence in weak operator topology and strong operator topology.

Theorem 2.1. [1] *Let $\{\varphi_n\}_{n\geq 1}$ and φ be analytic self-maps of the unit disk such that $\|\varphi_n\|_\infty < 1$, and let $\{\psi_n\}_{n\geq 1}$ and ψ be elements in H^2. Assume that each D_{ψ_n,φ_n} is bounded, and that $D_{\psi,\varphi}$ is a bounded nonzero operator on H^2. Then D_{ψ_n,φ_n} converges to $D_{\psi,\varphi}$ in weak operator topology if and only if*
(a) ψ_n converges weakly to ψ in H^2,
(b) φ_n converges weakly to φ in H^2,
(c) $\sup_n \|D_{\psi_n,\varphi_n}\| < \infty$.

Theorem 2.2. [1] *Let $\{\varphi_n\}_{n\geq 1}$ and φ be analytic self-maps of the unit disk such that $\|\varphi_n\|_\infty < 1$, and let $\{\psi_n\}_{n\geq 1}$ and ψ be elements in H^2 where ψ is nonzero. Assume that each D_{ψ_n,φ_n} and $D_{\psi,\varphi}$ are bounded operators on H^2 where $D_{\psi,\varphi}$ is nonzero. Then D_{ψ_n,φ_n} converges to $D_{\psi,\varphi}$ in strong operator topology if and only if*
(a) ψ_n converges to ψ in H^2,
(b) φ_n converges to φ in H^2,
(c) $\sup_n \|D_{\psi_n,\varphi_n}\| < \infty$.

The following result whose proof will not be given here complements the above theorems. We use the notation $H_0^2 = \{f \in H^2 : f' \in H^2\}$.

Theorem 2.3. *Let $\{\varphi_n\}_{n\geq 1}$ and φ be analytic self-maps of the unit disk such that* $\sup \|\varphi_n\|_\infty < 1$*, and let $\{\psi_n\}_{n\geq 1}$ and ψ be elements in H_0^2 where ψ is nonzero and bounded. If D_{ψ_n,φ_n} converges to $D_{\psi,\varphi}$ in strong operator topology of H_0^2, then D_{ψ_n,φ_n} converges to $D_{\psi,\varphi}$ in uniform operator topology of H_0^2.*

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Oral Presentation **$*$: Speaker**

GENERALIZED HERMITE-HADAMARD INEQUALITY ON SEMISPHERE

ALI BARANI*, NASSER ABBASI

Department of Mathematics, Faculty of Science, Lorestan University , Khoramabad, Iran
barani.a@lu.ac.ir; abasi.n@lu.ac.ir

Abstract. In this paper we investigate the notions of $P-$convex and strongly $P-$convex functions defined on convex subsets of unit semisphere $\mathbb{R}^3$. Some versions of Hermite-Hadamard inequality are given in this setting.

1. Introduction

The Hermite-Hadamard inequality for a convex function $f : I \to \mathbb{R}$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

$I \subset \mathbb{R}$, has received renewed attention by many authors [3]. Many particular cases in several variables have been investigated by S.S. Dragomir in [5, 6]. Some improvements of 1.1 are studied in [4, 8]. The study of convex set and functions in semisphere, has several more accurate results and applications (see [9, 10]). Let us recall some of notions and results from differential geometry often used in what follows, see [1, 7] and references therein. A subset S of the unit sphere $S^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$, called convex if any two points $x,\ y \in S$ can be joined by a unique minimizing geodesic whose image belongs to S. Let S be a nonempty convex subset of

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S^2. A function $f : S \to \mathbb{R}$ is said to be quasiconvex if for every x, $y \in S$ and every $t \in [0, 1]$,

$$f(\gamma(t)) \leq \max\{f(x), f(y)\}, \tag{1.2}$$

where $\gamma : [0, 1] \to S$ is the unique minimal geodesic in S with $\gamma(0) = x$ and $\gamma(1) = y$.

For every $p, q \in S^2$ with $\omega := \arccos\langle p, q\rangle = d(p, q) < \pi($ d is called the intrinsic distance or Riemannian distance on S^2), the unique minimal geodesic in S^2 joining p and q is given by the following formula

$$\gamma(t) = \frac{\sin((1-t)\omega)}{\sin\omega}p + \frac{\sin(t\omega)}{\sin\omega}q,\ \ t \in [0, 1]. \tag{1.3}$$

Let $M \subseteq \mathbb{R}^3$ be a 2-surface and $f : M \to \mathbb{R}$ be an integrable function. If $F : D \to M$ is a C^1 parametrization of M, where D is an open subset of $\mathbb{R}^2$ in $uv-$plane then, the surface integral of f on $R := F(D) \subseteq M$ is defined by

$$\int_R f ds := \int\int_D f(F(u, v))\|\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}\|dudv.$$

Note that, the surface integral does not depend on parametrization. Recall the following result from [2].

Lemma 1.1. *Let $0 < \omega_0 < \pi$. Then, for every $0 \leq \theta < 2\pi$ the curve*

$$\alpha_\theta(\varphi) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi),\ \ \varphi \in [0, \omega_0],$$

is the unique minimal geodesic from $\tilde{p} := (0, 0, 1)$ to

$$q := (\sin\omega_0\cos\theta, \sin\omega_0\sin\theta, \cos\omega_0).$$

The Hermite-Hadamard inequality for a convex function on semisphere is investigated in [2]. Our goal in this paper is to establish an analogue of the Hermite-Hadamard inequality for $P-$convex and strongly $P-$convex functions defined on semisphere of S^2.

2. $P-$convexity and Hermite-Hadamard inequality

In this section the Hermite-Hadamard inequality for $P-$convex and strongly convex functions on hemispheres is considered.

Definition 2.1. Let S be a nonempty convex subset of S^2 and $f : S \to \mathbb{R}^+$ be a real valued function, $\mathbb{R}^+ := [0, +\infty]$. Then,
(i) f is said to be $P-$convex (or belong to the class $P(I)$) if it is nonnegative and for every $x, y \in S$ and every $t \in [0, 1]$,

$$f(\gamma(t)) \leq f(x) + f(y), \tag{2.1}$$

(ii) f is said to be strongly $P-$convex if it is nonnegative and there exists a constant $c > 0$ such that for every $x, y \in S$ and every $t \in (0, 1)$,

$$f(\gamma(t)) \leq f(x) + f(y) - ct(1-t)d^2(x, y), \tag{2.2}$$

where $\gamma : [0, 1] \to S$ is the unique minimal geodesic in S with $\gamma(0) = x$ and $\gamma(1) = y$.

It is easy to see that $P(I)$ contain all non-negative convex and quasiconvex functions defined on proper convex subsets of sphere are $P-$convex. In the following example we introduce a $P-$convex function defined on a convex subset of S^2 which is not quasiconvex.

Example 2.2. Define the non-negative function $f : C \to \mathbb{R}$ as

$$f(x) := 2\varphi_0^2 - d^2(x, \tilde{p}),$$

where $C := B(\bar{p}, \varphi_0)$, $0 < \varphi_0 < \pi/2$. Then, f is a $P-$convex function C which is not quasiconvex on C.

Now we are in a position to establish the Hermite-Hadamard inequality for $P-$convex functions defined on the semispheres of S^2.

Theorem 2.3. *Let that $f : C \to \mathbb{R}$ be a $P-$convex integrable function. Then, the following inequalitiy holds*

$$f(\tilde{p}) \leq \frac{2}{area(C)} \int_C f ds \leq 2f(\tilde{p}) + \frac{1}{\pi \sin \varphi_0} \int_{\partial C} f(\sigma(\tau)) d\tau, \tag{2.3}$$

where σ is the parametrization of ∂C by arc length and $C := B(\bar{p}, \varphi_0)$.

Next result is an improvement of lemma 2.2 in[2] for strongly $P-$convex functions.

Theorem 2.4. *Let S be a convex subset of S^2 and $q \in S^2$. Suppose that $f : S \to \mathbb{R}$ is a real valued function. Then, f is strongly $P-$convex on S with constant $\lambda > 0$ if and only if for every $x \in S$ the function $z \mapsto f(z) - \lambda d^2(z, x)$ is $P-$convex on S.*

The following establish a version of Hermite-Hadamard inequality for strongly $P-$convex functions.

Theorem 2.5. *Let that $f : C \to \mathbb{R}$ be a strongly $P-$convex integrable function with constant $\lambda > 0$. Then, the following inequalitiy holds*

$$g(\tilde{p}) \leq \frac{2}{area(C)} \int_C g ds \leq 2g(\tilde{p}) + \frac{1}{\pi \sin \varphi_0} \int_{\partial C} g(\sigma(\tau)) d\tau, \tag{2.4}$$

where σ is the parametrization of ∂C by arc length, $g(z) := f(z) - \lambda d^2(z, x)$ and $C := B(\bar{p}, \varphi_0)$..

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GENERAL VERSION OF THE SANDOR'S INEQUALITY

BAYAZ DARABY, MOHAMMAD REZA KARIMZADEH*

Department of Mathematics, University of Maragheh, P. O. Box 55136-553, Maragheh, Iran
bdaraby@maragheh.ac.ir; mkmk0165@gmail.com

Abstract. In this paper, we show Sandor type inequality for pseudo-integrals. Indeed, we state classic version of this inequality for pseudo-integrals. Some illustrate examples are given for theorems.

1. Introduction

The theory of fuzzy measures and fuzzy integral (Sugeno integral) has introduced by Sugeno [6] in his Ph.D. theses on 1974. From 2007, some authors have studied on some others fuzzy integral inequalities. Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subseteq [-\infty, +\infty]$ endowed with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$. Recently, Daraby et al. generalized Stolarsky, Hardy and Feng Qi type inequalities for pseudo-integrals ([2, 3, 4]).
Sandor's inequality in classical case is the following form.

Theorem 1.1. [1] *Let $f : [a, b] \to \mathbb{R}$ be a convex and non-negative function. Then*

$$\frac{1}{b-a}\int_a^b f^2(x)dx \leq \frac{1}{3}\left[f^2(a) + f(a)f(b) + f^2(b)\right], \tag{1.1}$$

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holds.

2. Preliminary

Now, we are going to review some well known results of pseudo-operations, pseudo-analysis and pseudo-additive measures and integrals in details.

Let $[a, b]$ be a closed (in some cases can be considered semi-closed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

Definition 2.1. The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$, i.e., for each $x \in [a, b], \mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b).

Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 2.2. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b], \mathbf{1} \odot x = x$.

We shall consider the semiring $([a, b], \oplus, \odot)$ for two important (with completely different behavior) cases. The first case is when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty)$, i.e., pseudo-operations are given with:

$$x \oplus y = g^{-1}(g(x) + g(x)) \qquad \text{and} \qquad x \odot y = g^{-1}(g(x)g(x)). \tag{2.1}$$

Then, the pseudo-integral for a function $f : [c, d] \to [a, b]$ reduces on the $g-$integral

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\left(\int_c^d g(f(x))dx\right). \tag{2.2}$$

The second class is when $x \oplus y = \max(x, y)$ and $x \odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [a, b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \sup_{x \in \mathbb{R}} \left(f(x) \odot \psi(x)\right),$$

where function ψ defines sup-measure m. We denote by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$m(A) = \text{ess sup}_{\mu}(x | x \in A) = \sup\left\{a | \mu(x | x \in A, x > a) > 0\right\}.$$

Theorem 2.3. *Let m be a* sup*-measure on $([0, \infty], \mathbb{B}[0, \infty])$, where $\mathbb{B}([0, \infty])$ is the Borel σ-algebra on $[0, \infty]$, $m(A) = \text{ess sup}_{\mu}(\psi(x)|x \in A)$, and $\psi : [0, \infty] \to [0, \infty]$ is a continuous function. Then for any pseudo-addition $\oplus$ with a generator g there exists a family m_λ of $\oplus_\lambda$-measure on $([0, \infty], \mathbb{B})$, where $\oplus_\lambda$ is a generated by g^λ (the function g of the power $\lambda, \lambda \in (0, \infty)$) such that $\lim\limits_{\lambda \to \infty} m_\lambda = m$.*

Theorem 2.4. *Let* $([0,\infty], \sup, \odot)$ *be a semiring , when* $\odot$ *is a generated with* g*, i.e., we have* $x \odot y = g^{-1}(g(x)g(y))$ *for every* $x, y \in (0,\infty)$*. Let* m *be the same as in Theorem 2.3, Then there exists a family* $\{m_\lambda\}$ *of* $\oplus_\lambda$ *-measures, where* $\oplus_\lambda$ *is a generated by* $g^\lambda, \lambda \in (0,\infty)$ *such that for every continuous function* $f : [0,\infty] \to [0,\infty]$,

$$\int^{\sup} f \odot dm = \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda = \lim_{\lambda\to\infty} (g^\lambda)^{-1} \left(\int g^\lambda(f(x))dx \right). \quad (2.3)$$

3. Main Results

In this section, we express Sandor's inequality for pseudo-integrals.

Theorem 3.1. *Let* $f : [a,b] \to [c,d]$ *be a continuous, convex and non-negative function and* $g : [c,d] \to [0,\infty)$ *be a continuous and increasing function. Then*

$$\left(\frac{1}{b-a}\right) g \left(\int_{[a,b]}^{\oplus} f_\odot^2(x)dx \right) \le \frac{1}{3} g\Big(\left[f_\odot^2(a) \oplus f(a) \odot f(b) \oplus f_\odot^2(b) \right] \Big), \quad (3.1)$$

holds.

Corollary 3.2. *Let* $f : [0,1] \to [c,d]$ *be a continuous, convex and non-negative function and* $g : [c,d] \to [0,\infty)$ *be a continuous and increasing function. Then*

$$\left(\frac{1}{b-a}\right) g \left(\int_{[0,1]}^{\oplus} f_\odot^2(x)dx \right) \le \frac{1}{3} g \left[f_\odot^2(0) \oplus f(0) \odot f(1) \oplus f_\odot^2(1) \right], \quad (3.2)$$

holds.

Example 3.3. Let f and g are defined from $[0,1]$ to $[0,1]$ by $f(x) = x^2$ and $g(x) = \sqrt{x}$. Then we have

$$\frac{1}{4} = \frac{1}{1-0} \int_{[0,1]}^{\oplus} f_\odot^2(x)dx \le \frac{1}{3} g \left[f_\odot^2(0) \oplus f(0) \odot f(1) \oplus f_\odot^2(1) \right] = \frac{1}{3}.$$

We can not remove the assumption g is increasing in Theorem 3.1. The following example shows this fact.

Example 3.4. Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$. Then we have

$$\frac{5}{6} = \frac{1}{1-0} \int_{[0,1]}^{\oplus} f_\odot^2(x)dx \not\le \frac{1}{3} g \left[f_\odot^2(0) \oplus f(0) \odot f(1) \oplus f_\odot^2(1) \right] = \frac{1}{3}.$$

Theorem 3.5. *Let* $f : [a,b] \to [a,b]$ *be a measurable comonotone function and* $([a,b], \sup, \odot)$ *be a simiring and* m *be the same as Theorems 2.3 and 2.4. If* g *is the continuous and increasing function, then the following inequality is holds*

$$\left(\frac{1}{b-a}\right) g \left(\int_{[a,b]}^{\sup} f_\odot^2(x)dx \right) \le \frac{1}{3} g \left[f_\odot^2(a) \oplus f(a) \odot f(b) \oplus f_\odot^2(b) \right], \quad (3.3)$$

holds.

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MAXIMUM WEIGHTED INDEPENDENT SET WITH UNCERTAIN WEIGHTS

MEHDI DJAHANGIRI

Department of Mathematics, Faculty of Basic Science University of Maragheh, Maragheh, Iran
djahangiri.mehdi@maragheh.ac.ir

Abstract. The uncertainty theory from the viewpoint of Liu is a new way to deal with problems which some of parameters are not determinate. Especially, this theory is based on experts belifes and by introducing a measure in these belifes tries to overcome to uncertainty. Maximum weighted independent set problem is a classic combinatorial optimization problem and has wide range of application such as scheduling. It is proved that this is an NP-hrad problem and for arbitary graph, there are only approximate algorithms. In this paper, we investigate this problem with indeterministic weights and obtain an equivalant deministic integer programming model. Considering the concept of uncertainty distribution of an uncertain variable, one models is constructed based on α-chance method.

1. Introduction

When a real-world problem is modeled, the data are usualy considered indeterminate. In these situations, probability theory, fuzzy theory and theory of belief functions, also referred to as evidence theory or were introduced but unfortunatelly, these theories are not cover all problems. Recently, Baoding Liu proposed an axiomatic basis of uncertainty theory in 2007 [4] and refined it [5] in 2010. In this theory, the belifes of experts have essential role. In this paper, First, an integer programming model is presented for the maximum

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weighted independent set problem and a summary of uncertainty theory is explained and one model is discussed to solve this problem when the weights are uncertain.

2. An Integer Programming Model

$$\begin{array}{ll} \max & \sum_{i\in V} w_i x_i \\ \text{s.t.} & x_i + x_j \leq 1 \quad \forall (i,j)\in E, \\ & x_i \in \{0,1\}. \end{array} \tag{2.1}$$

As already mentioned previously, the problem (2.1) is an NP-hard and no algorithm can find a solution in polynomial time unless P=NP. For obtaining an approximate solution of this model, semidefinite or linear relaxation is utilized. In the next section, a summary of the of uncertainty theory will be expressed from the viewpoint of Liu.

3. Uncertainty Theory

As already mentioned, uncertainty theory can be a potential tool for expressing experts' beliefs in mathematical language and using them. In this section, we point out some important concepts and features of this theory. For more details, refer the reader to [5].

Let Γ be a nonempty set and $\mathcal{L}$ be a σ-algebra over it . Then $(\Gamma,\mathcal{L})$ is called measurable space and each member $\Lambda \in \mathcal{L}$ is called a measurable set or an *event*. Measurable space $(\Gamma,\mathcal{L})$ with uncertain measure $\mathcal{M}$ (this concept will be introduced later) is saied uncertainty space and is shown by $(\Gamma,\mathcal{L},\mathcal{M})$. A set function $\mathcal{M}$ over $\mathcal{L}$ is said to be an *uncertain measure* if it satisfies the following four axioms:

Axiom1 : (*Normality*) $\mathcal{M}\{\Gamma\}$=1 for the universal set Γ.

Axiom2 : (*Duality*) $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}$=1 for any event Λ.

Axiom3 : (*Subadditivity*) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leq \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

Axiom4 : (*Product*) Let $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ be uncertainty spaces for integer $k \geq 1$. The product uncertain measure $\mathcal{M}$ is the one satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$ respectively and $\bigwedge$ stands for the minimum operator.

The function $f : (\Gamma,\mathcal{L},\mathcal{M}) \to \mathbb{R}$ is said to be measurable if for any Borel set B of real numbers, it holds $f^{-1}(B) = \{\gamma | f(\gamma) \in B\} \in \mathcal{L}$. An *uncertain variable* ξ is a measurable function on an uncertainty space. Also,

ξ is considered nonegative if $\mathcal{M}\{\xi < 0\} = 0$ and positive if $\mathcal{M}\{\xi \leq 0\} = 0$. The next theorem talks about a fundamental and practical property in uncertainty theory.

Theorem 3.1. [5] *Let $\xi_1, \xi_2, \ldots, \xi_n$ be uncertain variables. Further, let f be a real valued measurable function. Then $f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable.*

For an uncertain variable ξ, *uncertainty distribution* Φ is defined as $\Phi(x) = \mathcal{M}\{\xi \leq x\}$. Different type of uncertain variables have been defined in the literature corresponding to different uncertainty distributions.

Definition 3.2. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be *independent* if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\},$$

for any Borel sets $B_1, B_2, \ldots, B_n$.

Theorem 3.3. [5] *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If f is a strictly increasing function, then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has the inverse uncertainty distribution*

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).$$

In some cases, validity of an equality is not determined and α-chance model can be a useful interpretation for such situations. It is said that an equality $g(x, \xi) \leq 0$ holds with the belief degree α when $\mathcal{M}\{g(x, \xi) \leq 0\} \geq \alpha$. Determining the feasible region associated to such constraints in higher dimensional spaces is not straightforward. Next theorem presents an equivalent crisp constraint in specific circumstances.

Theorem 3.4. [5] *Let $g(x, \xi_1, \xi_2, \ldots, \xi_n)$ be a strictly increasing function with respect to $\xi_1, \ldots, \xi_k$, and strictly decreasing with respect to $\xi_{k+1}, \ldots, \xi_n$. Further, let $\xi_1, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \ldots, \Phi_n$, respectively. Then the relation $\mathcal{M}\{g(x, \xi_1, \xi_2, \ldots, \xi_n) \leq 0\} \geq \alpha$ holds if and only if*

$$g(x, \Phi_1^{-1}(\alpha), \ldots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)) \leq 0.$$

In this section, the model (2.1) is investigated when it's wheights are uncertain variables ξ_i.

$$\begin{array}{ll} \max & \sum_{i \in V} \xi_i x_i \\ \text{s.t.} & x_i + x_j \leq 1 \quad \forall (i,j) \in E, \\ & x_i \in \{0,1\}. \end{array} \implies \begin{array}{ll} \max & t \\ \text{s.t.} & t \leq \sum_{i \in V} \xi_i x_i \\ & x_i + x_j \leq 1 \qquad \forall (i,j) \in E, \\ & x_i \in \{0,1\}. \end{array} \tag{3.1}$$

$$\begin{array}{ll} \max & t \\ \text{s.t.} & \mathcal{M}\{t \le \sum_{i\in V} \xi_i x_i\} \ge \alpha \\ & x_i + x_j \le 1 \ \forall (i,j) \in E, \\ & x_i \in \{0,1\}. \end{array} \quad \Longrightarrow \quad \begin{array}{ll} \max & t \\ \text{s.t.} & t \le \sum_{i\in V} \Phi_i^{-1}(\alpha) x_i \\ & x_i + x_j \le 1 \ \forall (i,j) \in E, \\ & x_i \in \{0,1\}. \end{array} \tag{3.2}$$

Finaly, by using theorem 3.3, the following deterministic model, is achieved.

$$\begin{array}{lll} \max & \sum_{i\in V} \Phi_i^{-1}(\alpha) x_i & \\ \text{s.t.} & x_i + x_j \le 1 & \forall (i,j) \in E, \\ & x_i \in \{0,1\}. & \end{array}$$

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Oral Presentation

SDP TECHNIQUE FOR THE TOTAL DOMINATING SET

MEHDI DJAHANGIRI

Department of Mathematics, Faculty of Basic Science University of Maragheh, Maragheh, Iran
djahangiri.mehdi@maragheh.ac.ir

Abstract. In this paper, one of the most famous NP-complete problems in graph theory, the total dominating set problem, was investigated and a new quadratic integer programming model was presented. Finally, an SDP relaxation models are proposed. Finding the efficiency of the relaxation could be a future research direction.

1. Introduction

Consider an undirected and connected graph $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ and E are respectively vertices and edges of G. The degree of vertex v_i is shown by $deg(v_i)$, and Δ stands for the maximum degree of the graph. A set $S \subseteq V$ is called dominating set of G if each vertex is a member of S or adjacent to a member of S. The set S is referred to as minimum dominating set if it has minimum cardinality among all dominating sets. The cardinality of minimum dominating set is called domination number and denoted by $\gamma(G)$. Domination number and its variations have been extensively studied in the literature. One of them is total domination number. A set S_t of vertices in a graph G is called a total dominating set if every vertex $v_i \in V$ is adjacent to an element of s_t. The size of total dominating set with minimum cardinality is denoted by $\gamma_t(G)$. For more details we refer the reader to [9].

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Dominating set and its variants are one of the classical problems in graph theory having important applications in many fields (e.g. [3, 4] for some recent applications). In [8], more than 1200 papers on different versions of dominating set problem are listed. Despite having a lot of application and theoretical attraction, Unfortunately, in [5] it has been shown the NP-completeness of dominaing set problem and subsequently the total dominating set problem. So, for any arbitary graph , it is not expected that the total dominating set will be found in reasonable time. To overcome to this challenge, there are several methods such linear relaxation, Greedy Algorithms and metaheuristics. In this paper, the semidefinite relaxation is applied to find an approximation solution for the total dominating set problem.
The semidefinite programming is a special case of convex optimization which linear objective function is optimized over the intersection of the cone of positive semidefinite matrices with linear constraints. Let $\mathbb{S}^n$ denote the set of symmetric $n \times n$ real matrices. The cone of symmetric positive semidefinite (definite) matrices is denoted by $\mathbb{S}^n_+$ ($\mathbb{S}^n_{++}$). $B-D \succeq 0$ ($B-D \succ 0$) means that $(B-D)$ is positive semidefinite (definite). Suppose that $A_1, \dots, A_m$ are linearly independent matrices in $\mathbb{S}^n$; $C \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$. The standard form of semidefinite programming problem is written as follows:

$$\begin{array}{ll} \min & \langle C, X\rangle \\ \text{s.t.} & \langle A_i, X\rangle \geq b_i \quad i = 1, 2, \dots, n \\ & X \succeq 0 \end{array}$$

where $\langle B, D\rangle = \operatorname{tr}(B^t D) = \sum_{i,j} b_{ij} d_{ij}$. The semidefinite programming model can be solved in a polynomial time with an interior point method [1]. The interested reader is referred to [2] for a thorough discussion and applications of semidefinite programming. semidefinite programming relaxation is a powerful tool to approximate the optimal solution of some combinatorial problems. For example, dominating set [6] and maximum cut [7]. The good performance of semidefinite relaxation in these problems encouraged us to utilize this method to find an approximation of the k-tuple domination number.

2. Problem Description

The open neighborhood of a vertex v consists of the set of adjacent vertices to v, that is, $N(v) = \{w \in V | wv \in E\}$ and the closed neighborhood of is defined as $N[v] = N(v) \cup \{v\}$. The following labelling can be defined on V with respect to a subset $S \subseteq V$ as:

$$y(v_i) = \begin{cases} 1 & v \in S \\ -1 & v \notin V \end{cases}$$

For the sake of simplicity, we denote $y(v_i)$ by y_i and refer to a vertex with the label 1 as (+1)-vertex and as (-1)-vertex, otherwise. Further, $N(i)(N[i])$ stands for the open (closed) neighborhood of the vertex v_i. It is important to mention that a vertex in a total dominating set S_t is a (+1)-vertex induced by S_t. From the definition of labelling, it is clear that the objective function is $\frac{1}{2}\sum_{i=1}^{n}(1+y_i)$. The next lemma gives us valid inequlities for total dominating set.

Lemma 2.1. *$S_t \subseteq V$ is total dominating set if and only if it must satisfy in the following inequalities:*

$$\sum_{j\in N(i)}(1-y_iy_j)+\sum_{j\in N[i]}\frac{1+y_j}{2}\geq 2 \qquad i=1,2,\dots,n. \tag{2.1}$$

Now, based on the (2.1), the quadratic integer programming model can be written as follows:

$$\begin{array}{lll} \min & \frac{1}{2}\sum_{i=1}^{n}(1+y_i) & \\ \text{s.t.} & \sum_{j\in N(i)}(1-y_iy_j)+\sum_{j\in N[i]}\frac{1+y_j}{2}\geq 2 & i=1,2,\dots,n \\ & y_i\in\{-1,+1\} & i=1,2,\dots,n \end{array} \tag{2.2}$$

Observe that the objective functions of (2.2) and part of inequalities are linear, while analyzing of our algorithms needs a quadratic objective function. To convert these linear functions to quadratic ones, a reference variable $y_0\in\{-1,+1\}$ is introduced and problem (2.2) is rephrased as follows:

$$\begin{array}{lll} \min & \frac{1}{2}\sum_{i=1}^{n}(1+y_0y_i) & \\ \text{s.t.} & \sum_{j\in N(i)}(y_0^2-y_iy_j)+\sum_{j\in N[i]}\frac{y_0^2+y_0y_j}{2}\geq 2 & i=1,2,\dots,n \\ & y_i\in\{-1,+1\} & i=0,1,2,\dots,n \end{array} \tag{2.3}$$

Now suppose $\overline{y}=(y_0,y_1,\dots y_n)$ be the optimal solution of (2.3). If $y_0=+1$ then $y=(y_1,\dots y_n)$ is the optimal solution of (2.2) and if $y_0=-1$ then $y=(-y_1,\dots -y_n)$ is the optimal solution of (2.2).

3. Semidefinite Relaxation

First, for $i=0,1,\dots,n$, the variable y_i is substituted by an $(n+1)$-dimensional vector $u_i\in\mathbb{U}$ where $\mathbb{U}=\{(+1,0,\dots,0),(-1,0,\dots,0)\}$. Accordingly, the restriction $y_i\in\{-1,+1\}$ is replaced by $u_i\in\mathbb{U}$ and then problem (2.3) is adapted as:

$$\begin{array}{lll} \min & \frac{1}{2}\sum_{i=1}^{n}(1+u_0^tu_i) & \\ \text{s.t.} & \sum_{j\in N(i)}(u_0^tu_0-u_i^tu_j)+\sum_{j\in N[i]}\frac{u_0^tu_0+u_0^tu_j}{2}\geq 2 & i=1,2,\dots,n \\ & u_i\mathbb{U} & i=0,1,2,\dots,n \end{array} \tag{3.1}$$

Recall that $||u_i=1||$ for $u_i\in\mathbb{U}$ and this motivates to expand $\mathbb{U}$ to the standard $(n+1)$-dimensional unit sphere $\mathcal{S}^{n+1}=\{u\in\mathbb{R}^{n+1}|\ ||u||=1\}$, at the second step of the relaxation procedure. Thus, the following problem is obtained

$$\begin{array}{lll} \min & \frac{1}{2}\sum_{i=1}^{n}(1+u_0^tu_i) & \\ \text{s.t.} & \sum_{j\in N(i)}(u_0^tu_0-u_i^tu_j)+\sum_{j\in N[i]}\frac{u_0^tu_0+u_0^tu_j}{2}\geq 2 & i=1,2,\dots,n \\ & u_i^tu_i=1, \quad u_i\in\mathcal{S}^{n+1} & i=0,1,2,\dots,n \end{array} \tag{3.2}$$

By introducing $X_{ij} = y_i y_j, E_{ij} = e_i e_j^t$ and $A_i = \sum_{j\in N(i)} \frac{1}{2}(2E_{00} - E_{ij} - Eji) + \sum_{j\in N[i]} \frac{1}{4}(2E_{00} - E_{0j} - Ej0)$, where e_i is the i-th standard unit vector of $\mathbb{R}^{n+1}$, the model (3.2) is converted to the following:

$$\begin{array}{lll} \min & \frac{n}{2} + \langle C, X\rangle & \\ \text{s.t.} & \langle A_i, X\rangle \geq 2 & i = 1, 2, \ldots, n \\ & X_{ii} = 1 & i = 0, 1, 2, \ldots, n \\ & \operatorname{rank}(X) = 1 & \\ & X \succeq 0 & \end{array} \tag{3.3}$$

where $C = (c_{ij}), c_{i0} = c_{0i} = \frac{1}{4}$ for $i = 1, \ldots, n$ and $c_{ij} = 0$ otherwise. By dropping the nonconvex constraint $\operatorname{rank}(X) = 1$ from (3.3), the semidefinite relaxation is formulated as:

$$\begin{array}{lll} \min & \frac{n}{2} + \langle C, X\rangle & \\ \text{s.t.} & \langle A_i, X\rangle \geq 2 & i = 1, 2, \ldots, n \\ & X_{ii} = 1 & i = 0, 1, 2, \ldots, n \\ & X \succeq 0 & \end{array} \tag{3.4}$$

The model (3.4) can be solved by interior point methods in CVX solver. Finally, the optimal solution of (3.4) just gives us a lower bound to total domination number.

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ON THE EXISTENCE AND UNIQUENESS OF PSEUDO ALMOST AUTOMORPHIC SOLUTIONS FOR INTEGRO DIFFERENTIAL EQUATIONS WITH REFLECTION

ZAHRA EIDINEJAD, REZA SAADATI*

School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 13114-16846, Iran
zahra_eidinejad@mathdep.iust.ac.ir; rsaadati@eml.cc; rsaadati@iust.ac.ir

Abstract. In this paper we apply fixed point theory and measure theory to investigate the existence of unique solutions for integro differential equations with reflection (IDE-R). Using almost automorphic functions, we study the solutions of these equations, which are of pseudo almost automorphic ($\mathcal{PAA}$) type, by introducing the Mittag-Leffler function. Finally, we present an example we illustrate the application of the main results obtained.

1. Introduction

In the extensive research on differential equations in the literature, different unique solutions such as periodic, almost periodic, and automorphic have been obtained for these equations and generalizations and ideas are presented in different fields (also researchers considered weighted pseudo almost periodic functions which is a generalization of pseudo almost periodicity functions) [1, 2].

The main purpose in this paper is to investigate the existence of solutions for the IDE-R, which is defined as follows (considering the continuous

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functions of $k, \varphi : \mathbb{R}^3 \to \mathbb{R}$ and $\mathcal{L} : \mathbb{R}^+ \to \mathbb{R}^+$):

$$\begin{aligned}\psi'(w) &= \tau\psi(w) + \kappa\psi(-w) + h(w) + k(w, \psi(\vartheta(w)), \psi(\vartheta(-w))) \\ &\quad + \int_{w}^{+\infty} \mathcal{L}(z-w)\varphi(z, \psi(\vartheta(w)), \psi(\vartheta(-w)))dw \\ &\quad + \int_{-w}^{+\infty} \mathcal{L}(z+w)\varphi(z, \psi(\vartheta(z)), \psi(\vartheta(-z)))dz, \;\; w \in \mathbb{R},\end{aligned} \tag{1.1}$$

for $\tau \in \mathbb{R}, \;\kappa \in \mathbb{R}^*$.

2. Basic concepts

Definition 2.1. For Banach space $\mathcal{Y}$ and every $k \in \mathcal{C}(\mathbb{R}, \mathcal{Y})$, suppose that (z_ℓ) is a real sequence. If there is a sub-sequence (z_{ℓ_k}) such that $\lim_{\ell_k\to\infty} k\left(r + z_{\ell_k}\right) = k(r)$ and $\lim_{\ell_k\to\infty} h\left(r - z_{\ell_k}\right) = k(r)$, then, k is said to be almost automorphic or $k \in \mathcal{AA}(\mathbb{R}, \mathcal{Y})$, for every $r \in \mathbb{R}$.

Definition 2.2 ([3])**.** We consider a ζ-field $\mathcal{Z}$ as type Lebesque of $\mathbb{R}$ and suppose $\mathcal{M}$ is the space of all positive measures on $\mathcal{Z}$. Then $\eta \in \mathcal{M}$ if

(*1*) $\eta([\tau, \kappa]) < \infty$, for all $\tau \le \kappa \in \mathbb{R}$,
(*2*) $\eta(\mathbb{R}) = +\infty$.

Definition 2.3 ([4])**.** Given the Banach space $\mathcal{Y}$ and the positive measure $\eta \in \mathcal{M}$, a function $k : \mathbb{R} \to \mathcal{Y}$ that is bounded continuous is called η-ergodic, $k \in \mathcal{E}(\mathbb{R}, \mathcal{Y}, \eta)$, if $\lim_{\mathtt{s}\to\infty} \frac{1}{\eta([-\mathtt{s},\mathtt{s}])} \int_{[-\mathtt{s},\mathtt{s}]} \|k(w)\| d\eta(w) = 0$, where $\eta([-\mathtt{s}, \mathtt{s}]) := \int_{-\mathtt{s}}^{\mathtt{s}} d\eta(r)$.

Definition 2.4 ([5])**.** Given the Banach space $\mathcal{Y}$ and the positive measure $\eta \in \mathcal{M}$, a function $k : \mathbb{R} \to \mathcal{Y}$ that is continuous is called η-$\mathcal{PAA}$ if $k = h_1 + \mathtt{u}_1$, where h_1 is an almost automorphic function ($h_1 \in \mathcal{AA}(\mathbb{R}, \mathcal{Y})$) and $\mathtt{u}_1$ is an ergodic function.

To prove the main results of this paper, we consider the following hypotheses:

($\boldsymbol{\mathcal{N}_1}$) There exist a continuous and increasing function $\vartheta : \mathbb{R} \to \mathbb{R}$ such that for all $v \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$, we have $vo\vartheta \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$.

($\boldsymbol{\mathcal{N}_2}$) For every $\gamma \in \mathbb{R}$, there exist $\vartheta > 0$ and a bounded interval J such that for positive measure η, we have $\eta(\{\tau + \gamma : \tau \in \mathcal{U}\}) \le \vartheta\eta(\mathcal{U})$, whenever $\mathcal{U} \in \mathcal{Z}$ satisfies $\mathcal{U} \cap J = \emptyset$.

($\boldsymbol{\mathcal{N}_3}$) There exist $\jmath, \ell > 0$ such that for all $\mathcal{U} \in \mathcal{Z}$,

$$\eta(-\mathcal{U}) \le \jmath + \ell\eta(\mathcal{U}).$$

($\boldsymbol{\mathcal{N}_4}$) There is a function $\rho : \mathbb{R} \to \mathbb{R}^+$ such that for all $\mathtt{E} \in \mathbb{B}(\mathbb{R})$, $\eta \in \mathcal{M}$ and $\eta_\vartheta(\mathtt{E}) = \eta\left(\vartheta^{-1}(\mathtt{E})\right)$ we have $d\eta_\vartheta(r) \le \rho(r)d\eta(r)$, ρ is also continuous, strictly increasing and

$$\limsup \frac{\eta[-\mathtt{P}(\mathtt{a}), \mathtt{P}(\mathtt{a})]}{\eta[-\mathtt{a}, \mathtt{a}]} \mathtt{Q}(\mathtt{P}(\mathtt{a})) < +\infty,$$

where $\mathtt{P}(\mathtt{a}) = |\vartheta(\mathtt{a})| + |\vartheta(-\mathtt{a})|$ and $\mathtt{Q}(\mathtt{P}(\mathtt{a})) = \sup_{r\in[-\mathtt{P}(\mathtt{a}),\mathtt{P}(\mathtt{a})]\rho(\mathtt{P})}$.

($\boldsymbol{\mathcal{N}_5}$) Given $\rho=\sqrt{\tau^2-\kappa^2}$, where $\tau>\kappa$, the following holds

$$\mathtt{D}_1(\rho,\eta):=\sup_{\mathtt{s}>0}\left\{\int_{-\mathtt{s}}^{\mathtt{s}}\sum_{\ell=0}^{\infty}\frac{(-\rho(r+\mathtt{s}))^{\ell}}{\Gamma(\ell\alpha+\theta)}d\eta(r)\right\}<\infty,$$

and

$$\mathtt{D}_2(\rho,\eta):=\sup_{\mathtt{s}>0}\left\{\int_{-\mathtt{s}}^{\mathtt{s}}\sum_{\ell=0}^{\infty}\frac{(-\rho(-r+\mathtt{s}))^{\ell}}{\Gamma(\ell\alpha+\theta)}d\eta(r)\right\}<\infty.$$

($\boldsymbol{\mathcal{N}_6}$) $k:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ has a Lipschitz coefficient $\mathrm{H}_k>0$ such that

$$|k(r,v_1,w_1)-k(r,v_2,w_2)|\leq \mathrm{H}_k\left(|v_1-v_2|+|w_1-w_2|\right),$$

for all $(v_1,w_1),(v_2,w_2)\in\mathbb{R}^2$.

($\boldsymbol{\mathcal{N}_7}$) $\varphi:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ has a Lipschitz coefficient $\mathrm{H}_\varphi>0$ such that

$$|\varphi(r,\psi_1,\psi_2)-\psi(r,F_1,F_2)|<\mathrm{H}_\varphi\left(|\psi_1-F_1|+|\psi_2-F_2|\right),$$

for all $\psi_1,\psi_2,F_1,F_2\in\mathbb{R}$.

($\boldsymbol{\mathcal{N}_8}$) There exists $\mathcal{L}:\mathbb{R}^+\to\mathbb{R}^+$ such that $g=\int_0^{+\infty}\mathcal{L}(w)dw<\infty$.

To prove the results we consider two states for the Lipschitz coefficients of the functions above. In one state (above) these coefficients are constant and in the second state (below) they are not constant. In the following, we express the necessary conditions according to the second state.

($\boldsymbol{\mathcal{N}_9}$) $k:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ has a Lipschitz function $\mathrm{H}_k\in \mathrm{L}^p(\mathbb{R},\mathbb{R},dv)\cap \mathrm{L}^p(\mathbb{R},\mathbb{R},d\eta)$ such that

$$|k(r,v_1,w_1)-k(r,v_2,w_2)|\leq \mathrm{H}_k(r)\left(|v_1-v_2|+|w_1-w_2|\right),$$

where $\eta\in\mathcal{M}$, $p>1$ and for all $(v_1,w_1),(v_2,w_2)\in\mathbb{R}^2$.

($\boldsymbol{\mathcal{N}_{10}}$) $\varphi:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ has a Lipschitz function $\mathrm{H}_\varphi\in \mathrm{L}^p(\mathbb{R},\mathbb{R},dv)\cap \mathrm{L}^p(\mathbb{R},\mathbb{R},d\eta)$ such that

$$|\varphi(r,v_1,w_1)-\varphi(r,v_2,w_2)|\leq \mathrm{H}_\varphi(r)\left(|v_1-v_2|+|w_1-w_2|\right),$$

where $\eta\in\mathcal{M}$, $p>1$ and for all $(v_1,w_1),(v_2,w_2)\in\mathbb{R}^2$.

($\boldsymbol{\mathcal{N}_{11}}$) There exists $\mathcal{L}:\mathbb{R}^+\to\mathbb{R}^+$, such that

$$\int_0^{+\infty}(\mathcal{L}(w))^{\gamma}dw<+\infty,\ \text{for all } \gamma>1.$$

3. Existence of a unique η-$\mathcal{PAA}$ solution for equation (1.1) in two-states

Theorem 3.1. *Let $k,\varphi\in\mathcal{PAP}(\mathbb{R},\mathbb{R},\eta)$ and assume that ($\boldsymbol{\mathcal{N}_4}$)-($\boldsymbol{\mathcal{N}_8}$), ($\boldsymbol{\mathcal{N}_1}$)-($\boldsymbol{\mathcal{N}_3}$) are satisfied. Then equation* (1.1) *has a unique η-$\mathcal{PAA}$ solution if*

$$\frac{|\rho-\tau|+|\rho+\tau|+2|\kappa|}{\rho\left(\sum_{\ell=0}^{\infty}\frac{(\rho r)^{\ell}}{\Gamma(\ell\alpha+\theta)}\sum_{\ell=0}^{\infty}\frac{(-\rho)^{\ell}r^{\ell+1}}{(\ell+1)\Gamma(\ell\alpha+\theta)}\right)}(\mathrm{H}_k+2g\mathrm{H}_\varphi)<1.$$

Theorem 3.2. *Consider $k, \varphi \in \mathcal{PAP}(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}, \eta)$ and assume that conditions ($\boldsymbol{\mathcal{N}_1}$)-($\boldsymbol{\mathcal{N}_5}$) and ($\boldsymbol{\mathcal{N}_9}$)-($\boldsymbol{\mathcal{N}_{10}}$) are satisfied. Then, equation (1.1) has a unique η-$\mathcal{PAA}$ solution if*

$$\|\mathrm{H}_k\|_{L^p(\mathbb{R},\mathbb{R},dv)} + 2\bigg(\int_0^{+\infty} (\mathcal{L}(w))^q\bigg)^{\frac{1}{q}} \|\mathrm{H}_\varphi\|_{L^p(\mathbb{R},\mathbb{R},dv)} < \frac{\sum_{\ell=0}^{\infty} \frac{((-\rho)^{\frac{1}{q}} r)^\ell}{\rho q^{\frac{1}{q}} \Gamma(\ell\alpha+\theta)} \sum_{\ell=0}^{\infty} \frac{(-\rho)^{\frac{\ell}{q}} r^{\ell+1}}{(\ell+1)\Gamma(\ell\alpha+\theta)}}{|\rho-\tau| + |\rho+\tau| + 2|\kappa|}, \tag{3.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Example 3.3. If we consider an equation of type (1.1) such that $\mathcal{L}(z) = \sum_{\ell=0}^{\infty} \frac{(-|z|)^\ell}{\Gamma(\alpha\ell+\theta)}$ for all $z \in \mathbb{R}^+$. In order for condition ($\boldsymbol{\mathcal{N}_1}$) to be true, we set $\vartheta(r) = \frac{1}{r+1} - e$. By putting $\tau = 4, \kappa = \sqrt{7}, \rho = \sqrt{\tau^2 - \kappa^2} = 3, k(r, v, w) = \varphi(r, v, w) = \frac{1}{15} \sum_{\ell=0}^{\infty} \frac{(-|r|)^\ell}{\Gamma(\alpha\ell+\theta)}[\sin v + \cos w]$, $p = q = \frac{1}{2}, \alpha = 1$, $\theta = 1$ and $\|\mathrm{H}_k\|_{L^2(\mathbb{R},\mathbb{R},dv)} = \|\mathrm{H}_\varphi\|_{L^2(\mathbb{R},\mathbb{R},dv)} = \frac{1}{15}$ and $\|\mathrm{H}_k\|_{L^2(\mathbb{R},\mathbb{R},d\eta)} = \|\mathrm{H}_\varphi\|_{L^2(\mathbb{R},\mathbb{R},d\eta)} \leq \frac{1}{15}\sqrt{\exp(1)}$ condition ($\boldsymbol{\mathcal{N}_7}$) is satisfiedn where $\mathrm{H}_k(r) = \mathrm{H}_\varphi(r) = \frac{1}{15} \sum_{\ell=0}^{\infty} \frac{(-|r|)^\ell}{\Gamma(\alpha\ell+\theta)}$. Therefore

$$\begin{aligned} &\|\mathrm{H}_k\|_{L^2(\mathbb{R},\mathbb{R},dv)} + 2\left(\int_0^{+\infty} (\mathcal{L}(w))^2 dw\right)^{\frac{1}{2}} \|\mathrm{H}_\varphi\|_{L^2(\mathbb{R},\mathbb{R},dv)} \\ &= \frac{\sqrt{2}+1}{15} < \frac{e^{18}}{|\rho-\tau| + |\rho+\tau| + 2|\kappa|} = \frac{e^{18}}{8 + 2\sqrt{7}}. \end{aligned}$$

all the conditions of theorem 3.2 are satisfied and equation has a unique η-$\mathcal{PAP}$ solution.

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Oral Presentation

RANGE OF IDEMPOTENT ADJOINTABLE OPERATORS ON HILBERT C^*-MODULES

R. ESKANDARI

Department of Mathematics, Farhangian University, Iran; rasoul.eskandari@cfu.ac.ir

Abstract. Let T and S be idempotent adjointable operators on the Hilbert C^*-module $\mathcal{H}$ over a C^*-algebra $\mathcal{A}$. We establish that if there exist constants $\alpha_1, \alpha_2 > 0$ such that for all $x \in R(T)$ and $y \in \mathcal{R}(S)$

$$|x+y| \geq \alpha_1 |x| \text{ and } |x+y| \geq \alpha_2 |y|,$$

then $\mathcal{R}(T) \cap \mathcal{R}(S) = \{0\}$ and $\mathcal{R}(T) + \mathcal{R}(S)$ is orthogonality complemented submodule of $\mathcal{H}$. We also show that if Π_1, Π_2 are idempotents in $\mathcal{L}(\mathcal{E})$ such that $\mathcal{R}(\Pi_1) \cap \mathcal{R}(\Pi_2) = \{0\}$ and $\mathcal{R}(\Pi_1) + \mathcal{R}(\Pi_2)$ is an orthogonally complemented submodule of $\mathcal{E}$, Then $\mathcal{R}(\Pi_1 + \Pi_2)$ is closed in $\mathcal{E}$ if and only if $\mathcal{R}(\Pi_1 - \Pi_2)$ is closed in $\mathcal{E}$.
Acknowledgment. This is a joint work with Professors W. Luo, M.S. Moslehian, Q. Xu, and H. Zhang.

1. Introduction

Let $\mathcal{A}$ be a C^*-algebra. A pre-Hilbert C^*-module $\mathcal{H}$ over $\mathcal{A}$ is a right $\mathcal{A}$-module equipped with a sesquilinear map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ satisfying:

(1) $\langle x, x \rangle \geq 0,\ x \in \mathcal{X}$; $\langle x, x \rangle = 0$ if and only if $x = 0$.
(2) $\langle x, y \rangle^* = \langle y, x \rangle,\ x, y \in \mathcal{X}$.
(3) $\langle x, ya \rangle = \langle x, y \rangle a,\ x, y \in \mathcal{X},\ a \in \mathcal{A}$.

If the norm defined by $\|x\|^2 = \|\langle x, x \rangle\|$ for all $x \in \mathcal{X}$ is complete we say $\mathcal{X}$ is a Hilbert C^*-module. Suppose that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert C^*-modules. Let

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$\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of all maps $T:\mathcal{H}\to\mathcal{K}$ for which there is an application $T^*:\mathcal{H}\to\mathcal{K}$ such that

$$\langle Tx,y\rangle=\langle x,T^*y\rangle,\quad x\in\mathcal{H},y\in\mathcal{K}. \tag{1.1}$$

With the abbreviation we denote by $\mathcal{L}(\mathcal{H},\mathcal{H})=\mathcal{L}(\mathcal{H})$. We denote by $\mathcal{R}(T)$ and $\mathcal{N}(T)$ the range and nullity of an operator T, respectively. Let $\mathcal{M}$ be a closed submodule of $\mathcal{X}$. Then we set

$$\mathcal{M}^{\perp}:=\{x\in\mathcal{X};\langle x,y\rangle=0,y\in\mathcal{M}\}\ .$$

We say that $\mathcal{M}$ is an orthogonally complemented submodule of $\mathcal{X}$ if $\mathcal{X}=\mathcal{M}+\mathcal{M}^{\perp}$. A closed submodule $\mathcal{M}$ is not necessarily orthogonally complemented. If $T\in\mathcal{L}(\mathcal{X})$ has closed range, it is known that $\mathcal{R}(T)$ and $\mathcal{N}(T)$ are orthogonally complemented. The study of the properties of Hilbert C^*-modules and also the investigation of the facts that have been established in the Hilbert space and their generalization to the Hilbert C^*-module have been of interest to mathematical researchers, for examples see [3, 5].
For each idempotent operator Π,

$$\mathcal{R}(\Pi)\cap\mathcal{R}(I-\Pi)=0\quad\text{and}\quad\mathcal{R}(\Pi)+\mathcal{R}(I-\Pi)=\mathcal{E}. \tag{1.2}$$

A problem is that if Π_1 and Π_2 are idempotent operator, then do we have $\overline{\mathcal{R}(\Pi_1)+\mathcal{R}(\Pi_2)}$ is orthogonally complemented submodule? In the Hilbert space case we have the classic criteria of closeness for the sum of a couple of subspaces.

Theorem 1.1. [4, Propsitin 2.1],[1, Theorem 13] *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be closed subspaces of $\mathcal{H}$. The following conditions are equivalent:*

(1) *$\mathcal{H}_1+\mathcal{H}_2$ is closed;*
(2) *$\|P_1P_2-P_{\mathcal{H}_1\cap\mathcal{H}_2}\|<1$;*
(3) *$\mathcal{H}_1^{\perp}+\mathcal{H}_2^{\perp}$ is closed;*
(4) *$\mathcal{R}((I-P_1)P_2)$ is closed;*
(5) *$\mathcal{R}(I-P_1P_2)$ is closed;*

2. Main Results

Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of a Hilbert space $\mathcal{H}$. Recall that the cosine of angle between $\mathcal{M}$ and $\mathcal{N}$ defined as follows:

$$c_0(\mathcal{M},\mathcal{N}):=\sup\big\{\|\langle x,y\rangle\|:x\in\mathcal{M},y\in\mathcal{N},\|x\|\le1,\|y\|\le1\big\}.$$

We have the following characterization in Hilbert spaces:

Theorem 2.1. [1, Theorem 12] *The following statements are equivalent.*

(1) *$c_0(\mathcal{M},\mathcal{N})<1$;*
(2) *$\mathcal{M}\cap\mathcal{N}=\{0\}$ and $\mathcal{M}+\mathcal{N}$ is closed;*
(3) *There exist a constant $\alpha>0$ such that*

$$\|x+y\|\ge\alpha_1\|x\|\quad(x\in\mathcal{M},y\in\mathcal{N}). \tag{2.1}$$

In [2], it is defined the *separated pair* of the closed submodules of a Hilbert C^*-modules.

Definition 2.2. Let $\mathcal{H}$ and $\mathcal{K}$ be closed submodules of $\mathcal{E}$. Then we say that $(\mathcal{H},\mathcal{K})$ is a *separated pair* if

$$\mathcal{H}\cap\mathcal{K}=0 \text{ and } \mathcal{H}+\mathcal{K} \text{ is orthogonally complemented in } \mathcal{E}. \tag{2.2}$$

Now we give the following result.

Theorem 2.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be orthogonally complemented closed submodules of $\mathcal{E}$. The following statements are equivalent:*

(i) *$(\mathcal{H},\mathcal{K})$ is a separated pair of orthogonally complemented submodules.*
(ii) *There are idempotents Π_1 and Π_2 in $\mathcal{L}(\mathcal{E})$ such that $\Pi_1\Pi_2=\Pi_2\Pi_1=0$, $\mathcal{R}(\Pi_1)=\mathcal{H}$ and $\mathcal{R}(\Pi_2)=\mathcal{K}$.*
(iii) *There is an idempotent $\Pi\in\mathcal{L}(\mathcal{E})$ such that $\mathcal{R}(\Pi)=\mathcal{H}$ and $\mathcal{K}\subseteq\mathcal{N}(\Pi)$.*

Corollary 2.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be orthogonally complemented closed submodules of $\mathcal{E}$. Then $(\mathcal{H},\mathcal{K})$ is a separated pair if and only if there exist constants $\alpha_1,\alpha_2>0$ such that $|x+y|\geq\alpha_1|x|$ and $|x+y|\geq\alpha_2|y| \quad (x\in\mathcal{H}, y\in\mathcal{K})$.*

Theorem 2.5. *Let Π_1,Π_2 be idempotents in $\mathcal{L}(\mathcal{E})$ such that $(\mathcal{R}(\Pi_1),\mathcal{R}(\Pi_2))$ is a separated pair of orthogonally complemented submodules of $\mathcal{E}$. Then $\mathcal{R}(\Pi_1+\Pi_2)$ is closed in $\mathcal{E}$ if and only if $\mathcal{R}(\Pi_1-\Pi_2)$ is closed in $\mathcal{E}$.*

The following example shows that the separation condition in Theorem 2.5 is necessary.

Example 2.6. Let $\mathcal{K}$ be a separable Hilbert space and let T be a non closed range operator on $\mathcal{K}$. Let $\mathcal{E}=\mathcal{K}\oplus\mathcal{K}$ and define idempotent operators Π_1 and Π_2 on $\mathcal{E}$ by

$$\Pi_1=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix},\quad \Pi_2=\begin{pmatrix} I & 0\\ T & 0\end{pmatrix}.$$

Note that $\mathcal{R}(\Pi_1)=\mathcal{K}\oplus 0$ and $\mathcal{R}(\Pi_2)=\{x\oplus Tx :\in\mathcal{K}\}$. Then $\mathcal{R}(\Pi_1)+\mathcal{R}(\Pi_2)=\mathcal{K}\oplus\mathcal{R}(T)$. This shows that $(\mathcal{R}(\Pi_1),\mathcal{R}(\Pi_2))$ is not separated pair of closed subspaces in $\mathcal{E}$. Since $\mathcal{R}(\Pi_1-\Pi_2)=0\oplus\mathcal{R}(T)$, so $\mathcal{R}(\Pi_1-\Pi_2)$ is not a closed subspace. Furthermore, the equation $\mathcal{R}(\Pi_1+\Pi_2)=\{2x\oplus Tx :\in\mathcal{K}\}$ yields that $\mathcal{R}(\Pi_1+\Pi_2)$ is a closed subspace in $\mathcal{E}$.

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MONOTONICITY AND CONVEXITY OF THE VALUE FUNCTION OF A FUZZY MINIMIZATION PROBLEM

SAMIRA FATEMI*, ILDAR SADEQI AND FRIDOUN MORADLOU

Department of Mathematics, Sahand University of Technology, Tabriz, Iran
s_fatemi98@sut.ac.ir; esadeqi@sut.ac.ir; moradlou@sut.ac.ir

Abstract. We consider a fuzzy minimization problem and we show that the value function associated with this problem is monotone and convex.

1. Introduction

Throughout this paper, $\mathbb{R}^m$ denotes the m-dimensional Euclidean space and $F(\mathbb{R})$ denotes the set of all fuzzy subsets on $\mathbb{R}$. A fuzzy subset of $\mathbb{R}$ is a function $u : \mathbb{R} \to [0, 1]$. For each such fuzzy set u and $r \in (0, 1]$, we denote r-level set of u by $[u]^r = \{x \in \mathbb{R} : u(x) \geq r\}$. the support of u is denoted by $suppu = \{x \in \mathbb{R} : u(x) \geq 0\}$ and the closure of the support of u is $[u]^0 = \overline{\{x \in \mathbb{R} : u(x) \geq 0\}}$. Suppose that $u \in F(\mathbb{R})$ satisfies the following conditions:

(1) u is normal; that is, there is an $x_0 \in \mathbb{R}$ with $u(x_0) = 1$;
(2) u is a convex fuzzy set; that is, $u((1 - r)x + ry) \geqslant \min(u(x), u(y))$ whenever $x, y \in \mathbb{R}$ and $r \in [0, 1]$;
(3) $u(x)$ is upper semi-continuous;
(4) $[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is a compact set,

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* Speaker.

Then u is called a fuzzy number. We use E to denote the fuzzy number space.
For any $a \in \mathbb{R}$, define a fuzzy number $\hat{a}$ by

$$\hat{a}(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{if } x \neq a. \end{cases}$$

Also

$$\hat{0}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0, \end{cases}$$

for any $x \in \mathbb{R}$.
The addition, scalar multiplication and multiplication on E is defined by

$$(u+v)(x) = \sup_{y+z=x} \min[u(y), v(z)],$$

$$(\lambda u)(x) = \begin{cases} u(\lambda^{-1}x) & \text{if } \lambda \neq 0, \\ \hat{0} & \text{if } \lambda = 0 \end{cases}$$

for $u, v \in E$, $\lambda \in \mathbb{R}$.
For any $u_i \in E$, $i = 1, 2, ..., n$, we call the ordered one-dimension fuzzy number class $u_1, u_2, ..., u_n$ (i.e., the Cartesian product of one-dimension fuzzy number $u_1, u_2, ..., u_n$) a n-dimension fuzzy vector, denote it as $(u_1, u_2, ..., u_n)$, and call the collection of all n-dimension fuzzy vectors (i.e., the Cartesian product $\overbrace{E \times E \times ... \times E}^{n}$) n-dimension fuzzy vector space, and denote it as $(E)^n$.

Definition 1.1. [1] Let $C \subseteq \mathbb{R}^m$ be a convex set. The fuzzy mapping $F : C \to E$ is convex if and only if

$$F((1-\lambda)x + \lambda y) \leq (1-\lambda)F(x) + \lambda F(y),$$

for every $x, y \in C$ and $\lambda \in [0, 1]$.

2. The Value Function

Definition 2.1. [1, 3] The matrix $A = [a_{ij}]_{p \times m}, i = 1, ..., p; j = 1, ..., m$, is a fuzzy matrix if its entries are fuzzy numbers, i.e. $a_{ij} \in E$. the operations on fuzzy matrices are defined by operations between fuzzy numbers.

Let $F : M \subseteq \mathbb{R}^m \to E$ be a fuzzy mapping and $G : M \subseteq \mathbb{R}^m \to (E)^k$ be k-dimensional fuzzy vector-valued function. Consider the fuzzy minimization problem

$$F_{opt} = \min_{x \in M} \{F(x) : G(x) \leq \hat{0}, Ax + B = \hat{0}\} \tag{2.1}$$

where $G(x) = (G_1(x), G_2(x), ..., G_k(x))$ and $G_1, G_2, ..., G_k : M \subseteq \mathbb{R}^m \to E$ are fuzzy number-valued functions and A, B are $p \times m$, $p \times 1$ fuzzy matrices, respectively.

The value function associated with problem (2.1) is the function $V : (E)^k \times (E)^p \to E$ (see [2]) given by

$$V(u, w) = \min_{x \in M}\{F(x) : G(x) \leq u, Ax + B = w\}. \tag{2.2}$$

We can write the feasible set of (2.2) as follows

$$C(u, w) = \{x \in M : G(x) \leq u, Ax + B = w\}. \tag{2.3}$$

The value function can be rewritten as

$$V(u, w) = \min\{F(x) : x \in C(u, w)\}. \tag{2.4}$$

3. Monotonicity and convexity of the value function

Theorem 3.1. *Let $F : M \subseteq \mathbb{R}^m \to E$ be a fuzzy mapping and $G : M \subseteq \mathbb{R}^m \to (E)^k$ be k-dimensional fuzzy vector-valued function, $M \subseteq \mathbb{R}^m$ a nonempty set and A, B are $p \times m$, $p \times 1$ fuzzy matrices, respectively. Let V be the value function given in (2.2). Then*

$$V(u, w) \geq V(t, w) \textit{ for any } u, t \in (E)^k, w \in (E)^p \textit{ satisfying } u \leq t.$$

Proof. Take $x \in C(u, w)$. Then from (2.3), we obtain that $G(x) \leq u \leq t$. Therefore, $x \in C(t, w)$. As a result $C(u, w) \subseteq C(t, w)$. Now, by (2.4) we obtain

$$V(u, w) \geq V(t, w).$$

□

Theorem 3.2. *Let $F : M \subseteq \mathbb{R}^m \to E$ and $G_1, G_2, ..., G_k : M \subseteq \mathbb{R}^m \to E$ be convex fuzzy-valued functions, $M \subseteq \mathbb{R}^m$ a nonempty convex set and A, B are $p \times m$, $p \times 1$ fuzzy matrices, respectively. Then V is convex over $(E)^k \times (E)^p$.*

Proof. Let $(u, w), (t, s) \in dom(v)$ and $\lambda \in [0, 1]$. To prove the convexity, we will show that

$$V(\lambda(u, w) + (1 - \lambda)(t, s)) \leq \lambda V(u, w) + (1 - \lambda)V(t, s).$$

i.e.

$$V(\lambda u + (1 - \lambda)t, \lambda w + (1 - \lambda)s) \leq \lambda V(u, w) + (1 - \lambda)V(t, s).$$

By the definition of the value function (2.2), sequences $(x_k) \in C(u, w), (y_k) \in C(t, s)$ exist such that

$$\lim_{k\to\infty} F(x_k) = V(u, w) \tag{3.1}$$

and

$$\lim_{k\to\infty} F(y_k) = V(t, s). \tag{3.2}$$

Since $(x_k) \in C(u, w)$ and $(y_k) \in C(t, s)$, we have $G(x_k) \leq u, G(y_k) \leq t$. Therefore, by the convexity of $G_1, G_2, ..., G_k$,

$$G(\lambda x_k + (1 - \lambda)y_k) \leq \lambda G(x_k) + (1 - \lambda)G(y_k) \leq \lambda u + (1 - \lambda)t. \tag{3.3}$$

Also,

$$A(\lambda x_k + (1 - \lambda)y_k) + b = \lambda(Ax_k + b) + (1 - \lambda)(Ay_k + b) = \lambda w + (1 - \lambda)s. \tag{3.4}$$

Combining (3.3) and (3.4), we conclude that

$$\lambda x_k + (1-\lambda)y_k \in C(\lambda u + (1-\lambda)t, \lambda w + (1-\lambda)s). \tag{3.5}$$

By the convexity of F,

$$F(\lambda x_k + (1-\lambda)y_k) \leq \lambda F(x_k) + (1-\lambda)F(y_k). \tag{3.6}$$

Now, According to (3.1), (3.2) and (3.6) we have

$$\liminf_{k\to\infty} F(\lambda x_k + (1-\lambda)y_k) \leq \lambda V(u,w) + (1-\lambda)V(t,s). \tag{3.7}$$

By (3.5) and the definition of V ((2.2)), for all k,

$$V(\lambda u + (1-\lambda)t, \lambda w + (1-\lambda)s) \leq F(\lambda x_k + (1-\lambda)y_k),$$

Therefore

$$V(\lambda u + (1-\lambda)t, \lambda w + (1-\lambda)s) \leq \liminf_{k\to\infty} F(\lambda x_k + (1-\lambda)y_k).$$

We combine the last inequality with (3.7) and obtain

$$V(\lambda u + (1-\lambda)t, \lambda w + (1-\lambda)s) \leq \lambda V(u,w) + (1-\lambda)V(t,s),$$

establishing the convexity of V. □

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Oral Presentation

MINIMAL INVARIANT PAIRS OF CYCLIC (NONCYCLIC) RELATIVELY NONEXPANSIVE MAPPINGS

MOOSA GABELEH

Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran
gab.moo@gmail.com, Gabeleh@abru.ac.ir

Abstract. We consider the class of cyclic (noncyclic) relatively nonexpansive mappings and study the structure of minimal invariant pairs in (strictly convex) Banach spaces. Then we conclude a well-known best proximity point (pair) theorem for such class of mappings.

1. Introduction

Let X be a Banach space and $C \subseteq X$. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is well known that if C is a nonempty, compact and convex subset of a Banach space X, then any nonexpansive mapping of C into C has a fixed point.

In the case that C is weakly compact and convex subset of a Banach space X, then the nonexpansive mapping T may not have a fixed point. If C possesses *normal structure*, then the existence of a fixed point is guaranteed by *Kirk's fixed point theorem* ([5]). We mention that every bounded, closed and convex subset of a *uniformly convex Banach space* X has normal structure.

Let A and B be two nonempty subsets of a normed linear space X. A mapping $T\colon A \cup B \to A \cup B$ is said to be *cyclic (noncyclic)* provided that $T(A) \subseteq B, T(B) \subseteq A$ ($T(A) \subseteq A, T(B) \subseteq B$).

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In the case that T is cyclic a point $x^\star \in A \cup B$ is called a *best proximity point* of T whenever

$$\|x^\star - Tx^\star\| = \operatorname{dist}(A,B) := \inf\{\|x-y\| \colon x \in A,\ y \in B\}.$$

Moreover, a point $(p,q) \in A \times B$ is said to be a *best proximity pair* of the noncyclic mapping T if

$$p = Tp, \quad q = Tq, \quad \|p-q\| = \operatorname{dist}(A,B).$$

A mapping $T\colon A \cup B \to A \cup B$ is said to be *relatively nonexpansive* if $\|Tx - Ty\| \leq \|x-y\|$ whenever $x \in A$ and $y \in B$.

To describe our results, we need some definitions and notations. We shall say that a pair (A,B) of subsets of a Banach space X satisfies a property if both A and B satisfy that property. For example, (A,B) is convex if and only if both A and B are convex; $(A,B) \subseteq (C,D) \Leftrightarrow A \subseteq C,$ and $B \subseteq D.$ We shall also adopt the notation

$$\delta_x(A) = \sup\{d(x,y) \colon y \in A\} \text{ for all } x \in X,$$
$$\delta(A,B) = \sup\{\delta_x(B) : x \in A\}.$$

The *closed and convex hull* of a set A will be denoted by $\overline{\text{con}}(A)$.

Definition 1.1. A Banach space X is said to be
(i) uniformly convex if there exists a strictly increasing function $\delta : (0,2] \to [0,1]$ such that the following implication holds for all $x,y,p \in X, R > 0$ and $r \in [0,2R]$:

$$\begin{cases} \|x-p\| \leq R, \\ \|y-p\| \leq R, \\ \|x-y\| \geq r \end{cases} \Rightarrow \|\frac{x+y}{2} - p\| \leq (1 - \delta(\frac{r}{R}))R;$$

(ii) strictly convex if the following implication holds for all $x,y,p \in X$ and $R > 0$:

$$\begin{cases} \|x-p\| \leq R, \\ \|y-p\| \leq R, \\ x \neq y \end{cases} \Rightarrow \|\frac{x+y}{2} - p\| < R.$$

Lemma 1.2. *([1]) Let (K_1,K_2) be a pair of nonempty subsets of a normed linear space X. Then*

$$\delta(K_1,K_2) = \delta(\overline{\text{con}}(K_1), \overline{\text{con}}(K_2)).$$

Given (A,B) a pair of nonempty subsets of a Banach space, then its proximal pair is the pair (A_0,B_0) given by

$$A_0 = \{x \in A \colon \|x-y'\| = \operatorname{dist}(A,B) \text{ for some } y' \in B\},$$
$$B_0 = \{y \in B \colon \|x'-y\| = \operatorname{dist}(A,B) \text{ for some } x' \in A\}.$$

Proximal pairs may be empty but, in particular, if A and B are nonempty weakly compact and convex then (A_0,B_0) is a nonempty weakly compact convex pair in X.

Definition 1.3. Let (A, B) be a nonempty pair in a Banach space X. Then (A, B) is said to be a proximinal pair if $A = A_0$ and $B = B_0$.

Definition 1.4. Let (A, B) be a nonempty pair of sets in a Banach space X. A point p in A (q in B) is said to be a diametral point with respect to B (w.r.t. A) if $\delta_p(B) = \delta(A, B)$ ($\delta_q(A) = \delta(A, B)$). A pair (p, q) in $A \times B$ is diametral if both points p and q are diametral.

2. Minimal invariant pairs of relatively nonexpansive mappings

The following lemmas play important roles in our coming discussions.

Lemma 2.1. [2] *Let (A, B) be a nonempty weakly compact convex pair of a Banach space X and let $T\colon A \cup B \to A \cup B$ be a cyclic (noncyclic) relatively nonexpansive mapping. Then there exists $(K_1, K_2) \subseteq (A_0, B_0) \subseteq (A, B)$ which is minimal with respect to being nonempty closed convex and T-invariant pair of subsets of (A, B) such that*

$$\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B).$$

Moreover, the pair (K_1, K_2) is proximinal.

<u>Notation</u>. Let (A, B) be a nonempty, weakly compact and convex pair in a Banach space X and suppose $T : A \cup B \to A \cup B$ is a cyclic (noncyclic) relatively nonexpansive mapping. By $\mathcal{M}_T(A, B)$ we denote the set of all nonempty, closed, convex, minimal and T-invariant pair $(K_1, K_2) \subseteq (A, B)$ such that $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$.

Lemma 2.2. (Lemma 3.1 of [3]) *Let (A, B) be a nonempty weakly compact convex pair in a Banach space X and $T\colon A \cup B \to A \cup B$ a cyclic relatively nonexpansive mapping. If $(K_1, K_2) \in \mathcal{M}_T(A, B)$, then each pair $(p, q) \in K_1 \times K_2$ with $\|p - q\| = \operatorname{dist}(A, B)$ contains a diametral point (with respect to (K_1, K_2)).*

Lemma 2.3. (Lemma 3.8 of [3]) *Let (A, B) be a nonempty weakly compact convex pair of a strictly convex Banach space X. Let $T\colon A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping. If $(K_1, K_2) \in \mathcal{M}_T(A, B)$, then each $(p, q) \in K_1 \times K_2$ with $\|p - q\| = \operatorname{dist}(A, B)$ is a diametral pair (with respect to (K_1, K_2)), that is,*

$$\delta_p(K_2) = \delta_q(K_1) = \delta(K_1, K_2).$$

3. Best proximity points (pairs)

Definition 3.1. Let (A, B) be a nonempty, weakly compact and convex pair in a Banach space X and $T : A \cup B \to A \cup B$ be a cyclic (noncyclic) relatively nonexpansive mapping. We say that the pair (A, B) has the H-property, if for any $(K_1, K_2) \in \mathcal{M}_T$,

$$\max\{\operatorname{diam}(K_1), \operatorname{diam}(K_2)\} \leq \delta(K_1, K_2).$$

It was announced in [4] that if (A,B) is a nonempty, bounded, closed and convex pair of subsets of a uniformly convex Banach space X and $T: A\cup B \to A\cup B$ is a cyclic (noncyclic) relatively nonexpansive mapping, then (A,B) has the H-property.

Definition 3.2. Suppose (A,B) is a nonempty, disjoint, weakly compact and convex pair in a Banach space X and $T: A\cup B \to A\cup B$ is a cyclic (noncyclic) relatively nonexpansive mapping such that (A,B) has the H-property. Define

$$\omega_T := \inf\{\frac{\max\{\mathrm{diam}(K_1), \mathrm{diam}(K_2)\}}{\delta(K_1,K_2)} : (K_1,K_2) \in \mathcal{M}_T\}.$$

It is clear that $\omega_T \in [0,1]$.

Proposition 3.3. ([4]) *Let (A,B) be a nonempty, disjoint, bounded, closed and convex pair of subsets of a uniformly convex Banach space X and $T: A\cup B \to A\cup B$ be a cyclic (noncyclic) relatively nonexpansive mapping. Then $\omega_T = 0$.*

Theorem 3.4. ([4]) *Let (A,B) be a nonempty, disjoint, weakly compact and convex pair of subsets of a Banach space X and $T: A\cup B \to A\cup B$ be a cyclic (noncyclic) relatively nonexpansive mapping. If $\omega_T = 0$, then has a best proximity point (pair).*

Corollary 3.5. ([2]) *Let (A,B) be a nonempty, disjoint, bounded, closed and convex pair of subsets of a uniformly convex Banach space X and $T: A\cup B \to A\cup B$ be a cyclic (noncyclic) relatively nonexpansive mapping. Then T has a best proximity point (pair).*

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Oral Presentation

ON ELEMENTS OF SECOND DUAL OF A HYPERGROUP ALGEBRA

ALI GHAFFARI

Department of Mathematics, Semnan University, P.O.Box 35195-363, Semnan, Iran
aghaffari@semnan.ac.ir

ABSTRACT. Let H be a locally compact hypergroup with left invariant Haar measure and let $L^p(H)$, $1 \leq p < \infty$, be the complex Lebesgue space associated with it. Let $L^\infty(H)$ be the set of all locally measurable functions that are bounded except on a locally null set, modulo functions that are zero locally a.e. Let $\mu \in M(H)$. We want to find out when $\mu F \in L^1(H)$ implies that $F \in L^1(H)$. Some necessary and sufficient conditions is found for a measure μ for which if $\mu F \in L^1(H)$ for every $F \in L^\infty(H)^*$, then $F \in L^1(H)$.

1. Introduction

Hypergroups are locally compact spaces whose bounded Radon measures form an algebra which has similar properties to the convolution measures algebra of a locally compact group. Locally compact hypergroups were independently introduced around the 1970's by Dunkl, Jewett and Spector. They generalize the concepts of locally compact groups with the purpose of doing standard harmonic analysis. For the theory of hypergroups and most of the basic properties we refer to [2], [4] and [5].
Let H be a locally compact Hausdorff space. Let $M(H)$ be the space of complex-valued, regular Borel measures on H. We denote by $M^1(H)$ the convex set formed by the probability measures on H. The support of a measure μ is denoted by suppμ. Let $\mathcal{C}(H)$ be the space of all compact

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subsets of H. The triple $(M(H), +, *)$ will be called a hypergroup if the following conditions are satisfied.

(1) the vector space $(M(H), +)$ admits a binary operation $*$ under which it is an algebra,
(2) for $x, y \in H$, $\delta_x * \delta_y$ is a probability measure on H with compact support,
(3) the mapping $(x, y) \mapsto \delta_x * \delta_y$ of $H \times H$ into $M(H)$ is continuous,
(4) the mapping $(x, y) \mapsto \mathrm{supp}(\delta_x * \delta_y) \in \mathcal{C}(H)$ is continuous with respect to the Michael topology on the space $\mathcal{C}(H)$ of nonvoid compact sets in H,
(5) there exists a unique $e \in H$ such that for every $x \in H$, $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$,
(6) there exists a necessarily unique involution (a homeomorphism $x \mapsto \tilde{x}$ of H onto itself with the property $(\tilde{x})\tilde{} = x$ for all $x \in H$) such that $(\delta_x * \delta_y)\tilde{} = \delta_{\tilde{y}} * \delta_{\tilde{x}}$,
(7) for $x, y \in H$, $e \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if $x = \tilde{y}$.

In the following we will write just H instead of $(M(H), +, *)$. It is still unknown if an arbitrary hypergroup admits a left Haar measure. It particular, it remains unknown whether every amenable hypergroup admits a left Haar measure. But all the known examples such as commutative hypergroups and central hypergroups do, for more information see [1] and [2]. In this case, one can define the convolution algebra $L^1(H)$ with multiplication $f * g(x) = \int f(x * y)g(\tilde{y})d\lambda(y)$ for all $f, g \in L^1(H)$. Recall that $L^1(H)$ is a Banach subalgebra and an ideal in $M(H)$ with a bounded approximate identity [2]. It should be noted that these algebras include not only the group algebra $L^1(G)$ but also most of the semigroup algebras.

2. Main results

Let H be a hypergroup with left Haar measure λ. The first Arens product on $L^\infty(H)^*$ is defined in stages as follows.
Let $\mu, \nu \in L^1(H)$, $f \in L^\infty(H)$ and $F, G \in L^\infty(H)^*$;

(i) Define $f\mu \in L^\infty(H)$ by $\langle f\mu, \nu\rangle = \langle f, \mu * \nu\rangle$;
(ii) Define $Ff \in L^\infty(H)$ by $\langle Ff, \mu\rangle = \langle F, f\mu\rangle$;
(iii) Define $GF \in L^\infty(H)^*$ by $\langle GF, f\rangle = \langle G, Ff\rangle$.

$L^\infty(H)^*$ is a Banach algebra, for more details see [3].

Theorem 2.1. *Let H be a hypergroup with left Haar measure λ. Then the following conditions are equivalent:*

(i) *there exists $0 \neq \mu \in L^1(H)$ such that if $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$, then $F \in L^1(H)$;*
(ii) *H is discrete.*

The following corollary is a direct consequence of theorem 2.1.

Corollary 2.2. *Let H be a compact hypergroup. Then the following conditions are equivalent:*

(i) *there exists $0 \neq \mu \in L^1(H)$ such that if $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$, then $F \in L^1(H)$;*
(ii) *H is finite.*

Let H be a compact hypergroup. Let $\mu \in L^1(H)$. The mapping $x \mapsto \delta_x * \mu$ is weakly continuous. Since H is compact, $\{\delta_x * \mu;\ x \in H\}$ is relatively weakly compact. By the Krein-Smulian theorem the closed, convex, circled hull of $\{\delta_x * \mu; x \in H\}$ is also weakly compact. It follows that $\{\nu * \mu; \nu \in L^1(H), \|\nu\| \leq 1\}$ is relatively weakly compact. It is easy to see that $\{\mu F; F \in L^\infty(H)^*, \|F\| \leq 1\}$ is relatively weakly compact. Suppose that $F \in \{F \in L^\infty(H)^*; \|F\| \leq 1\}$ and $\{\nu_\alpha\}$ is a net in $\{\nu \in L^1(H); \|\nu\| \leq 1\}$ which converges to F in the weak*-topology. Therefore $\{\mu * \nu_\alpha\}$ converges to μF in the weak*-topology. Passing to a subnet if necessary, we can assume that $\{\mu * \nu_\alpha\}$ converges weak to a measure $\nu \in L^1(H)$. Consequently $\mu F = \nu \in L^1(H)$.
The next corollary is an immediate consequence of above explanation.

Corollary 2.3. *Let H be an infinite compact hypergroup. Then $L^1(H)$ is a right ideal in $L^\infty(H)$ and $L^1(H)$ is not reflexive.*

Theorem 2.4. *Let H be a hypergroup with left Haar measure λ. The following two properties of an element μ in $M(H)$ are equivalent:*

(i) *if $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$, then $F \in L^1(H)$;*
(ii) *if $\{\nu_n\}$ is a bounded sequence in $L^1(H)$ such that $\{\mu * \nu_n\}$ is weakly convergent, then $\{\nu_n\}$ contains a weakly convergent subsequence.*

For a non-empty subset S of $L^1(H)$. The annihilator of S, denoted $Ann(S)$, is the set of all elements ν in $L^1(H)$ such that, for all μ in S, $\mu * \nu = 0$. In set notation,

$$Ann(S) = \{\nu \in L^1(H); \mu * \nu = 0 \ \text{ for all } \mu \in S\}.$$

Proposition 2.5. *Let H be a hypergroup with left Haar measure λ. The following two properties of an element μ in $M(H)$ are equivalent:*

(i) *if $F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$, then $F \in L^1(H)$;*
(ii) *$Ann(\mu)$ is reflexive and $\overline{\mu B_S} \subseteq \mu S$ for every closed subspace S of $L^1(H)$.*

Let $\mathbb{C}$ be the multiplicative group of all complex numbers. Let $\mu \in M(\mathbb{C})$. Consider the following assertions:

(i) if $\mu F \in L^1(\mathbb{C})$, then $F \in L^1(\mathbb{C})$;
(ii) $\nu \in M(\mathbb{C})$ and $\mu * \nu \in L^1(\mathbb{C})$ imply $\nu \in L^1(\mathbb{C})$.

Clearly (i) impies (ii). Are the converse implication true?

Proposition 2.6. *Assume that H is a commutative hypergroup. Let $\mu \in M(H)$, and let $\{\mu F; F \in L^\infty(H)^*\} + L^1(H)$ be a dense subspace of $L^\infty(H)^*$. If $\mu F \in L^1(H)$, then $F \in L^1(H)$.*

Proof. Let $F \in L^\infty(H)^*$ such that $\mu F \in L^1(H)$. Let $G \in L^\infty(H)^*$ and $\{\nu_\alpha\}$ be a net in $L^1(H)$ such that $\nu_\alpha \to G$ in the weak*-topology [3]. We can write

$$\mu FG = \lim_\alpha \mu F \nu_\alpha = \lim_\alpha \nu_\alpha * \mu F = G\mu F,$$

because H is commutative. This shows that $\mu FG = G\mu F$ for all $G \in L^\infty(H)^*$. Fix $G \in L^\infty(H)^*$. By assumption, $\{\mu F; F \in L^\infty(H)^*\} + L^1(H)$ is a dense subspace of $L^\infty(H)^*$. Consequently, we can find sequences $\{F_n\} \subseteq L^\infty(H)^*$ and $\{\mu_n\} \subseteq L^1(H)$ with $\{\mu F_n + \mu_n\}$ norm-convergent to G. Therefore

$$\begin{aligned} FG &= \lim_n F(\mu F_n + \mu_n) = \lim_n F\mu F_n + F\mu_n \\ &= \lim_n \mu F_n F + \mu_n F = \lim_n (\mu F_n + \mu_n)F = GF. \end{aligned}$$

Therefore $FG = GF$ for all $G \in L^\infty(H)^*$. We next show that $F \in Z_t(L^\infty(H)^*) = L^1(H)$ [2]. Indeed, if $\{G_\alpha\}$ is a net in $L^\infty(H)^*$ and $G_\alpha \to G$ in the weak*-topology, then

$$\begin{aligned} \lim_\alpha \langle FG_\alpha, f\rangle &= \lim_\alpha \langle G_\alpha F, f\rangle = \lim_\alpha \langle G_\alpha, Ff\rangle \\ &= \langle G, Ff\rangle = \langle GF, f\rangle, \end{aligned}$$

for all $f \in L^\infty(H)$. On the other hand, $\langle GF, f\rangle = \langle FG, f\rangle$. Hence $FG_\alpha \to FG$ (in the weak*-topology) implies that F is in the topological center of $L^\infty(H)^*$. This completes our proof. □

Recall that a basic sequence $\{x_n\}$ in a Banach space X is said to be boundedly complete if for each sequence of scalars $\{\alpha_n\}$, $\sum_{n=1}^\infty \alpha_n x_n$ is convergent whenever $\sup\{\|\sum_{i=1}^n \alpha_i x_i\|; n \in \mathbb{N}\} < \infty$.

Proposition 2.7. *Let H be a hypergroup with a left Haar measure, and let $\mu \in M(H)$. Consider the following assertions:*

(i) *If $\{\mu_n\}$ is a basic sequence in B and $\sum_{i=1}^\infty \|\mu * \mu_n\| < \infty$, then $\{\mu_n\}$ is boundedly complete;*

(ii) *$F \in L^\infty(H)^*$ and $\mu F \in L^1(H)$ imply $F \in L^1(H)$.*

Then the implication $(i) \to (ii)$ hold.

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Oral Presentation **$*$: Speaker**

GRADIENT ESTIMATE FOR $\Delta u + au(\log u)^p + bu = f$ UNDER THE RICCI SOLITON CONDITION

SAKINEH HAJIAGHASI* AND SHAHROUD AZAMI

Department of Pure Mathematics, Imam Khomeini International University, Qazvin, Iran
s.hajiaghasi@edu.ikiu.ac.ir; azami@sci.ikiu.ac.ir

Abstract. In this paper, we consider the gradient estimate of the following equation

$$\Delta u + au(\log u)^p + bu = f,$$

for some smooth function f and real constants a, b and we obtain an upper bound for gradient of u. As an application we obtain the gradient bound for the Riemannian manifold with Bakry-Émery Ricci curvature.

1. Introduction

Let (M, g) be an n-dimensional Riemannian manifold with a fixed base point $O \in M$. Consider the following lower bound on the Ricci curvature

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g \geq -\lambda g, \tag{1.1}$$

for some constant $\lambda \geq 0$ and smooth vector field X which satisfies the following condition:

$$|X|(y) \leq \frac{K}{d(y, O)^{\alpha}}, \qquad \forall y \in M. \tag{1.2}$$

Here $d(y, O)$ reperesent the distance from O to y, K is a positive real constant, and $0 \leq \alpha < 1$.

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* Speaker.

In the pioneering work of Zhang and Zhu [3], they proposed following main conditions (1.1) and (1.2) on a Riemannian manifold and moreover the following volume noncollapsing condition

$$Vol(B(x,r)) \geq \rho, \tag{1.3}$$

for all $x \in M$ and some constant $\rho > 0$. Based on this assumptions they obtained Volume comparison theorem, Isoperimetric inequality and Sobolev inequality which leaded to the Elliptic and Parabolic gradient estimates.
Gradient estimate for the solutions of the Poisson equation and heat equation are very powerful tools in geometry and analysis. As an important application Li and Yau [1] deduced a Harnack inequality and also they obtained upper and lower bounds for heat kernel under the Dirichlet and Neumann boundary conditions. Recently Peng .et.al [2], stablished Yau-type gradient estimates for following equation on Riemannian manifolds

$$\Delta u + au(\log u)^p + bu = 0,$$

where $a, b \in \mathbb{R}$ and p is a rational number with $p = \dfrac{k_1}{2k_2+1} \geq 2$, where k_1 and k_2 are positive integer numbers.
In this paper, using the sufficient instrument like Sobolev inequality and Volume comparison Theorem from [3] and with the same method we want to obtain gradient estimate for the smooth function u which satisfies

$$\Delta u + au(\log u)^p + bu = f, \tag{1.4}$$

here a, b and $p > 0$ are real constant and $f : M \to \mathbb{R}$ is a smooth function.

2. MAIN RESULTS

We may use following isoperimetric and sobolev inequality. The proof process is just like [**?**], we can prove the theorem for any $r \leq r_0 = r_0(n, K_1, K, \alpha, \rho)$.

Theorem 2.1 (Isoperimetric inequality)**.** *Let M be a Riemannian manifold equiped with the Ricci soliton which next three conditions hold on it.*

$$Ric + \frac{1}{2}\mathcal{L}_X g \geq -\lambda g, \qquad |X|(y) \leq \frac{K}{d(y,O)^\alpha}, \quad Vol(B(x,1)) \geq \rho,$$

for all $x \in M$ and some constant $\rho > 0$ and $K \geq 0$. (we could just have the first two equations when $\alpha = 0$). Then there is a constant $r_0 = r_0(n, K_1, K, \alpha, \rho)$ such that for any $r \leq r_0$ and $f \in C_0^\infty(B(x,r))$, we have

$$\mathrm{ID}_n^*(B(x,r)) \leq C(n)r.$$

Here $\mathrm{ID}_n^*(B(x,r))$ is the isoperimetric constant defined by

$$\mathrm{ID}_n^*(B(x,r)) = Vol(B(x,r))^{\frac{1}{n}} . \sup_{\Omega} \left\{ \frac{Vol(\Omega)^{\frac{n-1}{n}}}{Vol(\partial\Omega)} \right\},$$

where the supremum is taken over all smooth domains $\Omega \subset B(x,r)$ with $\partial\Omega \cap \partial B(x,r) = \emptyset$.

Theorem 2.2 (Sobolev inequality). *Under the same conditions as in the above theorem, we have the following Sobolev inequalities.*

$$\left(\oint_{B(x,r)} |f|^{\frac{n}{n-1}} dg\right)^{\frac{n-1}{n}} \leq C(n) r \oint_{B(x,r)} |\nabla f| dg,$$

and

$$\left(\oint_{B(x,r)} |f|^{\frac{2n}{n-2}} dg\right)^{\frac{n-2}{n}} \leq C(n) r^2 \oint_{B(x,r)} |\nabla f|^2 dg.$$

Moreover, for the case that $X = \nabla f$ for some smooth function f, we get

$$\left(\oint_{B(x,r)} |f|^{\frac{n}{n-1}} dg\right)^{\frac{n-1}{n}} \leq C(n) r \oint_{B(x,r)} |\nabla f| dg.$$

Theorem 2.3 (Volume comparison). *Assume that for an n-dimension Riemannian manifold, (1.1) and (1.2) hold. Suppose in addition that the volume non-collapsing condition holds*

$$Vol(B(x,1)) \geq \rho,$$

for positive constants $\rho > 0$, $K \geq 0$ and $0 \leq \alpha < 1$, then for any $0 < r_1 < r_2 \leq 1$, we have the volume ratio bound as follows

$$\frac{Vol(B(x,r_2))}{r_2^n} \leq e^{C(n,K_1,K,\alpha,\rho)[K_1(r_2^2 - r_1^2) + K(r_2 - r_1)^{1-\alpha}]} . \frac{Vol(B(x,r_1))}{r_1^n}. \quad (2.1)$$

In particular, this result are true by considering the gradient soliton vector field $V = \nabla f$.

Here is our main result:

Theorem 2.4. *Suppose that on a Riemannian manifold M^n, (1.1), (1.2) and (1.3) hold. For $q > \frac{n}{2}$, if u and f be smooth functions such that (1.4) holds with $0 \leq u \leq l_1$ and $|(\log u)^p| \leq l_2$ for constants l_1, l_2, then there exists a positive constant $r_0 = r_0(n, N, K, \alpha, \rho, l_1, l_2, l_3)$ such that for any $x \in M$ and $0 < r \leq r_0$ we have*

$$\sup_{B(x,\frac{1}{2}r)} |\nabla u|^2 \leq C(n, \lambda, K, \alpha, \rho, l_1, l_2)\left[(\|f\|^*_{2q,B(x,r)})^2 + r^{-2}(\|u\|^*_{2,B(x,r)})^2\right].$$

As an application we conclude:

Corollary 2.5. *Suppose that the following condition holds for a gradient Ricci soliton*

$$\mathrm{Ric} + Hessh \geq -\lambda g,$$

and more over we have two condition for potential function h as follows

$$|h(y) - h(z)| \leq K_1 d(y,z)^\alpha, \text{ and } \sup_{x \in M, 0 \leq r \leq 1} \left(r^\beta \|\nabla h\|^*_{q,B(x,r)}\right) \leq K_2.$$

Then there is a constant $r_0 = r_0(n, \lambda, K_1, K_2, \alpha, \beta, l_1, l_2)$*, such that by the same conditions as last theorem, the solution of (1.4) satisfies*

$$\sup_{B(x,\frac{r}{2})} |\nabla u|^2 \leq C(n, \lambda, K_1, K_2, \alpha, \beta, l_1, l_2)\left[r^{-2}(\|u\|^*_{2,B(x,r)})^2 + (\|h\|^*_{2q,B(x,r)})^2\right],$$

for any $q > \dfrac{n}{2}$.

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Oral Presentation $*$: **Speaker**

THE LOWER BOUND OF THE FIRST EIGENVALUES FOR A QUASILINEAR OPERATOR

SAKINEH HAJIAGHASI* AND SHAHROUD AZAMI

Department of Pure Mathematics, Imam Khomeini International University, Qazvin, Iran
s.hajiaghasi@edu.ikiu.ac.ir; azami@sci.ikiu.ac.ir

Abstract. In this paper, we use a special smooth function $f : \Omega \to \mathbb{R}$ on a bounded domain of a Riemannian manifold to estimate the lower bound of the first eigenvalue for quasilinear operator $Lf = -\Delta_p f + V|f|^{p-2}f$.

1. Introduction

It is well known that studying the eigenvalues and eigenfunctions of the Laplacian play an important role in global differential geometry since they reveal important relations between geometry of the manifold and analysis. So far, there have been some progress on the geometric operator as bi-Laplace, p-Laplace, and (p,q)-Laplace associated to a Riemannian metric g on a compact Riemannian manifold M^n. For instance, Lichnerowicz-type estimate had been studied in some research papers for the p-Laplace [4], p-Laplace with integral curvature condition [5], and recently investigated for the first eigenvalue of buckling and clamped plate problems in [3].
In this paper, we are going to study the first eigenvalue of following quasilinear operator which was introduced in [1], studied under considering different bounded Ricci curvature.

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Let (M^n, g, dv) be a compact Riemannian manifold with volume element dv, the quasilinear operator on M defines as

$$Lf = -\Delta_p f + V|f|^{p-2}f. \tag{1.1}$$

Here V is a nonnegative smooth function on M, and for $p \in (1, \infty)$ the p-Laplace operator is defined as

$$\Delta_p f = \mathrm{div}(|\nabla f|^{p-2}\nabla f).$$

Corresponding to the p-Laplacian we have the following eigenvalue equation

$$\begin{cases} Lf = \mu |f|^{p-2} f & on \quad M \\ f = 0 \ \ (Dirichlet) \ \ or \ \ \dfrac{\partial f}{\partial \nu} = 0 \ \ (Neumann) & on \quad \partial M \end{cases}$$

where ν is the outward normal on ∂M. The first nontrivial Dirichlet eigenvalue for M is given by

$$\mu_{1,p}(M) = \inf_{f \in W_0^{1,p}(M), f \neq 0} \frac{\int_M (|\nabla f|^p + V|f|^p) dv}{\int_M |f|^p dv},$$

and the first Neumann eigenvalue is given by

$$\lambda_{1,p}(M) = \inf_{f \in W^{1,p}(M), f \neq 0} \left\{ \frac{\int_M (|\nabla f|^p + V|f|^p) dv}{\int_M |f|^p dv}; \int_M |f|^{p-2} f dv = 0 \right\}.$$

Here $W^{1,p}(M)$ is the Sobolev space and $W_0^{1,p}(M)$ is the closure of $C_0^\infty(M)$ in Sobolev space $W^{1,p}(M)$. The function f is then called the eigenfunction of operator L corresponding to μ (or λ) on M.

2. MAIN RESULTS

We consider a bounded domain Ω in a n-dimensional Riemannian manifold M, $n \geq 2$. Under some boundary assumption for $f : \Omega \to \mathbb{R}$, we will obtain a positive lower bound for $\mu_{1,p}$ on bounded domain Ω as follows:

Theorem 2.1. *Let Ω be a bounded domain on a Riemannian manifold M, and assume that there is a smooth function $f : \Omega \to \mathbb{R}$ such that satisfies $\|\nabla f\| \leq a$ and $\Delta_p f \geq b$ for some positive constants a, b, where $a > b$. Then the first Dirichlet eigenvalue of the quasilinear operator L satisfies*

$$\mu_{1,p} \geq \frac{b^p}{p^p a^{p(p-1)}}.$$

Proof. We first note that by density we can use smooth functions in the variational characterization of $\mu_{1,p}(\Omega)$. So given $u \in C_0^\infty(M)$, based on the fact that V is positive function, we have

$$\begin{aligned} b\int_{\Omega}|u|^p dv &\leq \int_{\Omega}|u|^p(\Delta_p f+V)dv \\ &= -\int_{\Omega} < \nabla |u|^p, \|\nabla f\|^{p-2}\nabla f > dv + \int_{\Omega}|u|^p V dv \\ &= -p\int_{\Omega}|u|^{p-1} < \nabla|u|, \|\nabla f\|^{p-2}\nabla f > dv + \int_{\Omega}|u|^p V dv \\ &\leq p\int_{\Omega}|u|^{p-1}\|\nabla u\|\|\nabla f\|^{p-1}dv + \int_{\Omega}|u|^p V dv \\ &\leq p\int_{\Omega}|u|^{p-1}a^{p-1}\|\nabla u\|dv + \int_{\Omega}|u|^p V dv. \end{aligned} \tag{2.1}$$

Now considering a constant $c > 0$ and using Young inequality, we obtain

$$\begin{aligned} |u|^{p-1}a^{p-1}\|\nabla u\| &\leq \frac{c^q|u|^{q(p-1)}}{q} + \frac{a^{p(p-1)\|\nabla u\|^p}}{pc^p} \\ &= \frac{(p-1)c^{p/(p-1)}|u|^p}{p} + \frac{a^{p(p-1)\|\nabla u\|^p}}{pc^p}. \end{aligned}$$

Therefore

$$p\int_{\Omega}|u|^{p-1}a^{p-1}\|\nabla u\| + \int_{\Omega}|u|^p V \leq (p-1)c^{p/(p-1)}|u|^p + \frac{a^{p(p-1)}\|\nabla u\|^p}{c^p} + \int_{\Omega}|u|^p V. \tag{2.2}$$

We could choose c so that $b-(p-1)c^{p/(p-1)} = \frac{b}{p}$, that is $c^p = \frac{b^{p-1}}{p^{p-1}}$. Hence with the statement in theorem $a > b$, we know

$$\frac{p^{p-1}a^{p(p-1)}}{b^{p-1}} > 1,$$

so, (2.1) and (2.2) lead to

$$\frac{b}{p}\int_{\Omega}|u|^p dv \leq \frac{p^{p-1}a^{p(p-1)}}{b^{p-1}}\Big(\int_{\Omega}\|\nabla u\|^p dv + \int_{\Omega}|u|^p V dv\Big).$$

Dividing both side to $\int_{\Omega}|u|^p$, completes the proof. □

As a first application, we apply this theorem for distance function:

Corollary 2.2. *Consider a bounded domain $\Omega \in M^n(c)$. If Ω is contained in a geodesic ball B_R, then*

$$\begin{aligned} \mu_{1,p}(\Omega) &\geq \frac{(n-1)^p(\sqrt{-c})^p}{p^p}\coth^p(\sqrt{-c}R), \quad if \;\; c < 0, \\ \mu_{1,p}(\Omega) &\geq \frac{(n-1)^p}{p^p R^p}, \qquad\qquad\qquad\quad if \;\; c = 0, \\ \mu_{1,p}(\Omega) &\geq \frac{(n-1)^p(\sqrt{c})^p}{p^p}\cot^p(\sqrt{c}R), \quad if \;\; c > 0. \end{aligned}$$

Note that in the above corollary $M^n(c)$ is the simply connected space form of constant sectional curvature c.

Corollary 2.3. *Let $M^n = \mathbb{R} \times N$ be a warped product Riemannian manifold endowed with the warped metric $ds^2 = dt^2 + e^{2\rho(t)} g_0$, such that the warped fuction satisfies $\rho^{'}(t) \geq \kappa > 0$, for some constant κ. Then the first Dirichlet eigenvalue of (1.1), satisfies in the following:*

$$\mu_{1,p}(M) \geq \frac{(n-1)^p}{p^p} \kappa^p.$$

Based on the studies in [2], this kind of estimate that we mentioned for warped product can be lifted for Riemannian manifolds which admite a Riemannian submersion over hyperbolic space.

Theorem 2.4. *Let $\tilde{M}^m$ be a complete Riemannian manifold that admits a Riemannian submersion $\pi : \tilde{M}^m \longrightarrow M^n = \mathbb{R} \times N$, where π is a surjective map. If the mean curvature of the fibers satisfy $\|H^{\mathcal{F}}\| \leq \alpha$, for some $\alpha < (n-1)\kappa^{1/p}$, then for the first Dirichlet eigenvalue of (1.1), we have*

$$\mu_{1,p}(M) \geq \frac{((n-1)^p \kappa - \alpha)^p}{p^p}.$$

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Oral Presentation

REMARKS ON A NEW PROOF OF IRRATIONALITY OF e AND SOME APPLICATIONS

MEHDI HASSANI

Mehdi Hassani, Department of Mathematics, University of Zanjan, University Blvd., 45371-38791, Zanjan, Iran
mehdi.hassani@znu.ac.ir

Abstract. In this paper we provide a new proof of irrationality of the number e, based on integral representation of alternating sum of permutations. We provide some applications concerning the number of derangements.

1. Introduction

The number e has several interesting properties [4, Sec. 1.3], [8] and a rich history [9]. The most well-known proof of irrationality of this constant is due to J. Fourier [3, pp. 340–341], which is based on the series representation $e = \sum_{k=0}^{\infty} 1/k!$ [11, Sec. 3.32]. Also, an alternative proof based on the series representation $1/e = \sum_{k=0}^{\infty}(-1)^k/k!$ is known [1, Thm. 1.11].

Recently, among our investigation of the alternating sum of permutations [6], we could to provide a new proof of irrationality of the number e. Our intention in writing this seminar note is to highlight this proof and some of its related applications in analytic enumeration. In Section 2 we review our results on computing some families of sums over permutations. In Section 3 we obtain irrationality of e based on a permutation-sum computation. In Section 4 we give a proof of this summation identity to complete our proof of the irrationality of e. In the last section we provide some applications concerning the number of derangements.

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Key words and phrases. integration by parts, permutation, derangement.

2. Sums over permutations

Let $C(n,j)$ denote the number of j-combinations of n objects, and $P(n,j)$ denote the number of j-permutations of n objects, counting the number of ways to choose an ordered selection of j items from a set of n items. Many summation identities concerning $C(n,j)$ can be found in the literature. For example, see [5, Sec. 0.15], [10, Sec. 2.3.4], and [12, pp. 343–355] for a list of 334 identities. In comparison, there are fewer summation identities concerning $P(n,j)$ in the literature. Motivated by this fact, in [6, Thm. 1] we proved the following result.

Theorem 2.1. *Let $a \geqslant 1$ be a fixed real. For any positive integer n let*

$$L_n(a) = \int_1^a \log^n t \, \mathrm{d}t. \tag{2.1}$$

Then, for any integer $n \geqslant 1$ and for $x \geqslant 0$, we have

$$\sum_{j=0}^{n} (-1)^j \, P(n,j) x^{n-j} = \frac{(-1)^n \, n! + L_n(\mathrm{e}^x)}{\mathrm{e}^x}.$$

Also, in [7, Thm. 1.3, Thm. 1.5] we proved the following results generalizing the above theorem.

Theorem 2.2. *Let a be fixed real number, and for any integer $n \geqslant 0$ let*

$$E_n(a) = \int_{-\infty}^{a} t^n \mathrm{e}^t \, \mathrm{d}t.$$

Then, for any integer $n \geqslant 0$ and for each real $x \neq 0$ we have

$$S_n(x) := \sum_{j=0}^{n} P(n,j) \, x^j = (-1)^n \; x^n \, \mathrm{e}^{\frac{1}{x}} \, E_n\left(-\frac{1}{x}\right). \tag{2.2}$$

Theorem 2.3. *Given any positive integer k, there exist computable polynomials $A_k(x)$ and $B_k(x)$ such that for each $n \geqslant 1$ and for $x \neq 0$ we have*

$$\sum_{j=0}^{n} j^k \, P(n,j) \, x^j = \frac{A_k(x)}{x^k} + \frac{B_k(x)}{x^k} \, S_n(x),$$

where $S_n(x)$ is defined in (2.2) and the polynomials $A_k(x)$ and $B_k(x)$ have the following properties:

- *The coefficients of $A_k(x)$ and $B_k(x)$ are at most in terms of n,*
- $\deg A_k(x) = k-1$ *and* $\deg B_k(x) = k$,
- $A_1(x) = 1$, $B_1(x) = nx - 1$
- *for $k \geqslant 1$ the polynomials $A_k(x)$ and $B_k(x)$ satisfy the simultaneous recurrence*

$$\begin{cases} A_{k+1}(x) = B_k(x) - kxA_k(x) + x^2 A_k'(x), \\ B_{k+1}(x) = (nx - kx - 1)\, B_k(x) + x^2 B_k'(x). \end{cases} \tag{2.3}$$

Also, we mention that the truth of Theorem 2.3 has a connection with the incomplete gamma function $\Gamma(\alpha, z)$, which is defined by

$$\Gamma(\alpha, z) = \int_z^\infty t^{\alpha-1} \mathrm{e}^{-t} \, \mathrm{d}t.$$

Note that $E_n(a) = (-1)^n \, \Gamma(n+1, -a)$. Thus, for any integer $n \geqslant 0$ and for each real $x \neq 0$ we have $S_n(x) = x^n \, \mathrm{e}^{\frac{1}{x}} \, \Gamma\left(n+1, \frac{1}{x}\right)$, and considering the relation (2.3) we obtain the following corollary.

Corollary 2.4. *With assumptions and notations of Theorem 2.3 we have*

$$\sum_{j=0}^{n} j^k \, P(n,j) \, x^j = \frac{A_k(x)}{x^k} + \frac{B_k(x)}{x^k} \, x^n \, \mathrm{e}^{\frac{1}{x}} \, \Gamma\left(n+1, \frac{1}{x}\right), \tag{2.4}$$

where $\Gamma(\alpha, z)$ denotes the incomplete gamma function.

3. A new proof of irrationality of e

Recalling (2.1), we let $L_n = L_n(\mathrm{e}) = \int_1^{\mathrm{e}} \log^n t \, \mathrm{d}t$. By letting $x = 1$ in (2.1) we obtain

$$\sum_{j=0}^{n} (-1)^j \, P(n,j) = \frac{(-1)^n \, n! + L_n}{\mathrm{e}}. \tag{3.1}$$

We observe that $0 \leqslant \log t \leqslant 1$ for $1 \leqslant t \leqslant \mathrm{e}$. Thus, $0 < L_n \leqslant \int_1^{\mathrm{e}} \mathrm{d}t = \mathrm{e} - 1 < \mathrm{e}$, which implies $0 < \frac{L_n}{\mathrm{e}} < 1$ for each positive integer n. Now we assume that e is rational, say $\mathrm{e} = \frac{\alpha}{\beta}$ for some positive integers α and β. The relation (3.1) with $n = \alpha$ gives $\sum_{j=0}^{\alpha} (-1)^j P(n,j) = (-1)^\alpha (\alpha-1)! \beta + \frac{L_\alpha}{\mathrm{e}}$, implying that $\frac{L_\alpha}{\mathrm{e}}$ is an integer, a contradiction.

4. Proof of (3.1)

Integration by parts implies that $\int \log^j t \, \mathrm{d}t = t \log^j t - j \int \log^{j-1} t \, \mathrm{d}t$. Thus, the recurrence $L_j = \mathrm{e} - j L_{j-1}$ holds for any integer $j \geqslant 1$. Multiplying both sides of this recurrence by $\frac{(-1)^j}{j!}$, we can rewrite it as follows

$$\frac{(-1)^j}{j!} L_j - \frac{(-1)^{j-1}}{(j-1)!} L_{j-1} = \frac{(-1)^j}{j!} \mathrm{e}.$$

Summing over $1 \leqslant j \leqslant n$ yields

$$\frac{(-1)^n}{n!} L_n - L_0 = \mathrm{e} \sum_{j=1}^{n} \frac{(-1)^j}{j!}.$$

Since $L_0 = \mathrm{e} - 1$, we deduce that

$$\frac{(-1)^n}{n!} L_n = -1 + \mathrm{e} \sum_{j=0}^{n} \frac{(-1)^j}{j!} = -1 + \mathrm{e} \sum_{j=0}^{n} \frac{(-1)^{-j}}{j!}.$$

We multiply both sides of this identity by $(-1)^n n!$ to get

$$L_n = (-1)^{n+1} n! + \mathrm{e} \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{j!}.$$

Note that the last sum actually is $\sum_{j=0}^{n} (-1)^j P(n,j)$, the left hand side of (3.1). This completes the proof.

5. Applications concerning the number of derangements

In combinatorial mathematics, a derangement is a permutation of the elements of a set, such that no element appears in its original position. As an application of (3.1), we can give an integral representation for D_n, the number of derangements on a set of cardinality n. We observe that the alternating sum at the left hand side of (3.1) and D_n are related as follows:

$$D_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!} = (-1)^n \, n! \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!} = (-1)^n \sum_{j=0}^{n} (-1)^j P(n,j).$$

Thus, by considering the relation (3.1), we obtain

$$D_n = \frac{n!}{\mathrm{e}} + (-1)^n \frac{L_n}{\mathrm{e}}, \tag{5.1}$$

for each integer $n \geqslant 1$. The relation (5.1) is true for $n = 0$, too. This relation provides an explicit integral representation for the difference $D_n - \frac{n!}{e}$. In [6, Thm. 2] we used this integral representation to compute the moments of this difference, as follows:

Theorem 5.1. *We have*

$$\sum_{n=1}^{\infty} \left(D_n - \frac{n!}{\mathrm{e}} \right) = -1 + \frac{1}{\mathrm{e}} + \frac{\mathrm{Ei}(2) - \mathrm{Ei}(1)}{\mathrm{e}^2} \approxeq -0.218114,$$

where $\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{\mathrm{e}^{-z}}{z} \, \mathrm{d}z$ *is the exponential integral function defined by the Cauchy principal value of the integral. Also,*

$$\sum_{n=1}^{\infty} \left(D_n - \frac{n!}{\mathrm{e}} \right)^2 = -\frac{(\mathrm{e}-1)^2}{\mathrm{e}^2} + \frac{4}{\mathrm{e}^2} \int_0^{\frac{1}{2}} h(z) \, \mathrm{d}z \approxeq 0.433113,$$

where

$$h(z) = \frac{\mathrm{e}^{2z}}{\sqrt{1-z^2}} \arctan \frac{z}{\sqrt{1-z^2}} + \frac{\mathrm{e}^{2-2z}}{\sqrt{2z-z^2}} \arctan \frac{z}{\sqrt{2z-z^2}}.$$

Moreover, letting $\mathbf{x} = (x_1, \ldots, x_k)$, *for each integer* $k \geqslant 1$ *the following multiple integral representation holds:*

$$\sum_{n=1}^{\infty} \left(D_n - \frac{n!}{\mathrm{e}} \right)^k = -\frac{(\mathrm{e}-1)^k}{\mathrm{e}^k} + \frac{1}{\mathrm{e}^k} \int_0^1 \cdots \int_0^1 \frac{\mathrm{e}^{x_1+\cdots+x_k}}{1-(-1)^k x_1 \cdots x_k} \, \mathrm{d}\mathbf{x}.$$

Considering the notion of asymptotic series [2, Sec. 1.5], due to Poincaré, in [6, Thm. 3] we used (5.1) to obtain an asymptotic series for L_n and consequently for D_n, as follows:

Theorem 5.2. *Given any positive integer r, for any integer $n \geqslant 1$ we have the asymptotic expansions*

$$\frac{L_n}{\mathrm{e}} = \sum_{k=1}^{r} (-1)^{k-1} \frac{B_k}{n^k} + O\left(\frac{1}{n^{r+1}}\right),$$

and

$$D_n = \frac{n!}{e} + \sum_{k=1}^{r} (-1)^{n+k-1} \frac{B_k}{n^k} + O\left(\frac{1}{n^{r+1}}\right),$$

where B_k denotes the k-th Bell number and the constant of O-term does not exceed B_{r+1} in both expansions.

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REMARKS ON THE TRUNCATED WALLIS PRODUCT

MEHDI HASSANI

Mehdi Hassani, Department of Mathematics, University of Zanjan, University Blvd., 45371-38791, Zanjan, Iran
mehdi.hassani@znu.ac.ir

Abstract. In this paper we study the truncated Wallis product, by showing that for each fixed integer $m \geqslant 1$, there exists computable constants $C'_1, \ldots, C'_m$, such that as $n \to \infty$,

$$\prod_{k=1}^{n} \frac{2k \cdot 2k}{(2k-1)(2k+1)} = \left(1 + \sum_{k=1}^{m} \frac{C'_k}{n^k}\right) \frac{\pi}{2} + O\left(\frac{1}{n^{m+1}}\right).$$

1. Introduction

The Wallis product for π obtained in 1655 by John Wallis and appeared one year later in his *Arithmetica Infinitorum* [13, p. 179] in the following form

$$\frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 9 \times 9 \times 11 \times 11 \times 13 \times 13 \times \cdots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times 8 \times 10 \times 10 \times 12 \times 12 \times 14 \times \cdots}.$$

See [9], [10, Chap. 3] and [11] for a detailed description of Wallis' work. In modern terminology and notation, the Wallis product for π reads as follows

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{2k \cdot 2k}{(2k-1)(2k+1)}. \tag{1.1}$$

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Several researchers have established interesting properties of (1.1), including new proofs, generalizations, inequalities and connection with the probability theory. See [1, 4, 5, 8, 12, 14] and the references given there.

Standard proofs of the Wallis product for π runs over the integration of the powers of sine or cosine functions (see for example [6, Sec. 9.18]). Letting for each positive integer n,

$$I_n := \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x,$$

integration by parts gives $I_n = \frac{n-1}{n} I_{n-2}$. Repeated using this recurrence relation, we deduce that

$$I_{2n} = \frac{\pi}{2} \prod_{k=1}^{n} \frac{2k-1}{2k}, \quad \text{and} \quad I_{2n+1} = \prod_{k=1}^{n} \frac{2k}{2k+1}.$$

Dividing I_{2n+1} by I_{2n} we get $\frac{\pi}{2} = \mathcal{W}_n \eta_n$, where $\mathcal{W}_n$ is the truncated Wallis product given by

$$\mathcal{W}_n = \prod_{k=1}^{n} \frac{2k \cdot 2k}{(2k-1)(2k+1)}, \tag{1.2}$$

and

$$\eta_n = \frac{I_{2n}}{I_{2n+1}}. \tag{1.3}$$

Since $0 \leqslant \sin x \leqslant 1$ for $0 \leqslant x \leqslant \frac{\pi}{2}$, we observe that the sequence $(I_n)_{n \geqslant 1}$ is strictly decreasing, and consequently $1 \leqslant \eta_n \leqslant 1 + \frac{1}{2n}$. Thus, $\eta_n \to 1$ as $n \to \infty$. Equivalently, $\mathcal{W}_n \to \frac{\pi}{2}$ as $n \to \infty$, implying (1.1).

In this note we are motivated by studying the truncated form of the Wallis product. Considering the notion of asymptotic series [3, Sec. 1.5] due to Poincaré, we obtain an asymptotic series for the factor η_n, as follows.

Theorem 1.1. *Let $m \geqslant 1$ be fixed integer. There exists computable constants $C_1, \ldots, C_m$ such that as $n \to \infty$,*

$$\eta_n = 1 + \sum_{k=1}^{m} \frac{C_k}{n^k} + O\left(\frac{1}{n^{m+1}}\right). \tag{1.4}$$

Therefore, we have

$$\frac{\pi}{2} = \left(1 + \sum_{k=1}^{m} \frac{C_k}{n^k}\right) \mathcal{W}_n + O\left(\frac{1}{n^{m+1}}\right). \tag{1.5}$$

Remark 1.2. The value of the coefficients C_k are given by

$$C_k = \sum_{j=0}^{k} G_j\left(\frac{3}{2}, 1\right) G_{k-j}\left(\frac{1}{2}, 1\right), \tag{1.6}$$

where

$$G_k(a,b) = \binom{a-b}{k} B_k^{(a-b+1)}(a), \tag{1.7}$$

with $B_n^{(\ell)}(x)$ denoting the generalized Bernoulli polynomials, defined for integers $\ell \geqslant 0$ by

$$\left(\frac{t}{\mathrm{e}^t - 1}\right)^{\ell} \mathrm{e}^{xt} = \sum_{n=0}^{\infty} \frac{B_n^{(\ell)}(x)}{n!}\, t^n, \qquad |t| < 2\pi.$$

By computation, we get $\eta_n = P\left(\frac{1}{n}\right) + O\left(\frac{1}{n^{11}}\right)$, where

$$\begin{aligned} P(t) = 1 + \frac{1}{4}\,t - \frac{3}{32}\,t^2 + \frac{3}{128}\,t^3 + \frac{3}{2048}\,t^4 - \frac{33}{8192}\,t^5 - \frac{39}{65536}\,t^6 \\ + \frac{699}{262144}\,t^7 + \frac{4323}{8388608}\,t^8 - \frac{120453}{33554432}\,t^9 - \frac{208749}{268435456}\,t^{10}. \end{aligned}$$

Corollary 1.3. *Let $m \geqslant 1$ be fixed integer, $\mathcal{W}_n$ defined by (1.2), and the constants $C'_1, \dots, C'_m$ defined by the recurrence*

$$\sum_{j=0}^{k} C_j C'_{k-j} = 0, \tag{1.8}$$

with the initial values $C_0 = C'_0 = 1$ and $C_1, \dots, C_m$ given in (1.6). Then, as $n \to \infty$ we have

$$\mathcal{W}_n = \left(1 + \sum_{k=1}^{m} \frac{C'_k}{n^k}\right) \frac{\pi}{2} + O\left(\frac{1}{n^{m+1}}\right).$$

Remark 1.4. By computation, we have $\mathcal{W}_n = Q\left(\frac{1}{n}\right)\frac{\pi}{2} + O\left(\frac{1}{n^{11}}\right)$, where

$$\begin{aligned} Q(t) = 1 - \frac{1}{4}\,t + \frac{5}{32}\,t^2 - \frac{11}{128}\,t^3 + \frac{83}{2048}\,t^4 - \frac{143}{8192}\,t^5 + \frac{625}{65536}\,t^6 \\ - \frac{1843}{262144}\,t^7 + \frac{24323}{8388608}\,t^8 + \frac{61477}{33554432}\,t^9 - \frac{14165}{268435456}\,t^{10}. \end{aligned}$$

2. Proofs

Proof of Theorem 1.1. The idea to obtain an asymptotic series for the factor η_n is relating it by the Euler gamma function [7, Eq. 5.2.1], which is defined for $\Re(z) > 0$ by

$$\Gamma(z) = \int_0^{\infty} \mathrm{e}^{-t} t^{z-1} \mathrm{d}t,$$

and by analytic continuation for $\Re(z) \leqslant 0$ with simple poles of residue $\frac{(-1)^n}{n!}$ at $z = -n$, with $n \in \mathbb{N}$. To make this connection, we use the notion of the Beta function $\mathrm{B}(a,b)$ [7, Eq. 5.12.1], which is defined for complex variables a and b with $\Re(a) > 0$ and $\Re(b) > 0$ as follows

$$\mathrm{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{2.1}$$

The following trigonometric integral representation [7, Eqs. 5.12.2] holds the Beta function

$$\int_0^{\frac{\pi}{2}} \sin^{2a-1} x \cos^{2b-1} x \, \mathrm{d}x = \frac{1}{2} B(a, b).$$

Here we let $a = \frac{z+1}{2}$ with $\Re(z) > -1$, and $b = \frac{1}{2}$. By using (2.1) we deduce that

$$\int_0^{\frac{\pi}{2}} \sin^z x \, \mathrm{d}x = \frac{\Gamma(\frac{1}{2})}{2} \frac{\Gamma\left(\frac{z+1}{2}\right)}{\Gamma\left(\frac{z}{2}+1\right)}.$$

We recall that the Wallis product (1.1) and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [7, Eq. 5.4.6] are the same [2]. Hence, for each complex number z with $\Re(z) > -1$, we get

$$\int_0^{\frac{\pi}{2}} \sin^z x \, \mathrm{d}x = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{z+1}{2}\right)}{\Gamma\left(\frac{z}{2}+1\right)}.$$

By using this identity, and recalling (1.3), we obtain

$$\eta_n = \frac{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)^2}.$$

Asymptotic expansion for the ratio of two gamma functions [7, Eqs. 5.11.13, 5.11.17, 24.16.1] asserts that for any complex constants a and b, if $z \to \infty$ in the sector $|\arg(z)| \leqslant \pi - \delta < \pi$, then

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \sum_{k=0}^{\infty} \frac{G_k(a,b)}{z^k},$$

where $G_k(a, b)$ is defined in (1.7). Considering the notion of asymptotic series [3, Sec. 1.5], due to Poincaré, we read the above as follows

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b} \left(\sum_{k=0}^{m} \frac{G_k(a,b)}{z^k} + O\left(\frac{1}{|z|^{m+1}}\right)\right), \tag{2.2}$$

where $m \geqslant 1$ is any fixed integer. By using (2.2) we obtain

$$\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(n+1)} = n^{\frac{1}{2}} \left(\sum_{k=0}^{m} \frac{G_k(\frac{3}{2},1)}{z^k} + O\left(\frac{1}{n^{m+1}}\right)\right),$$

and

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} = n^{-\frac{1}{2}} \left(\sum_{k=0}^{m} \frac{G_k(\frac{1}{2},1)}{z^k} + O\left(\frac{1}{n^{m+1}}\right)\right).$$

Note that $G_0(a, b) = 1$ [7, Eqs. 5.11.15]. Thus, multiplying the above expansions gives (1.4) with C_k as in (1.6). This completes the proof. ☐

Proof of Corollary 1.3. By using the relation (1.5) we deduce that

$$\mathcal{W}_n = \left(1 + \sum_{k=1}^{m} \frac{C_k}{n^k}\right)^{-1} \frac{\pi}{2} + O\left(\frac{1}{n^{m+1}}\right).$$

We consider the Taylor expansion of the function $t \mapsto (1+t)^{-1}$ as $t \to 0$, and we let $t = \sum_{k=1}^{m} \frac{C_k}{n^k}$, where as assumed $n \to \infty$. Thus, there exists the constants $C'_1, \ldots, C'_m$ such that

$$\left(1 + \sum_{k=1}^{m} \frac{C_k}{n^k}\right)^{-1} = 1 + \sum_{k=1}^{m} \frac{C'_k}{n^k} + O\left(\frac{1}{n^{m+1}}\right),$$

or equivalently

$$\left(1 + \sum_{k=1}^{m} \frac{C_k}{n^k}\right)\left(1 + \sum_{k=1}^{m} \frac{C'_k}{n^k}\right) = 1 + O\left(\frac{1}{n^{m+1}}\right).$$

Comparing the coefficients of the both sides, we observe that the recurrence (1.8) holds for each k with $1 \leqslant k \leqslant m$. This completes the proof. $\square$

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Oral Presentation **$*$: Speaker**

HIGHER ORDER EXPONENTIALLY ISOMETRIC OPERATORS

KARIM HEDAYATIAN AND MARYAM SALEHI*

Department of Mathematics, College of Sciences, Shiraz University, Shiraz 7146713565, Iran
hedayati@shirazu.ac.ir, k.salehi@shirazu.ac.ir

Abstract. For a positive integer m, a bounded linear operator T on a Hilbert space is called an exponentially m-isometric operator if $\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}e^{kT^*}e^{kT}=0$. We show that for every non-empty compact subset K of pure imaginary axis, there exits an exponentially m-isometric operator T whose spectrum is K. Moreover, if $(T_n)_{n\geq 1}$ is a sequence of operators in this class that converges to T in the strong operator topology, then T is also an exponentially m-isometric operator.

1. Introduction

Throughout the paper, H stands for a Hilbert space and $B(H)$ denotes the space of all bounded linear operators on H. For a positive integer m, an operator $T \in B(H)$ is called an m-isometry if it satisfies the operator equation

$$\beta_m(T) := \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k = 0$$

where T^* denotes the adjoint operator of T. Since the pioneer work of Agler [1], the study of m-isometries has become an active area of research

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* Speaker.

in operator theory. Their applications to differential operator, disconjugacy and Brownian motion have been discussed in [2]. For more investigation on m-isometric operators one can see [3, 6].

An operator T is called an exponentially m-isometry if $\exp T$ is an m-isometric operator. Exponentially 1-isometric operators are simply exponentially isometries. The set of all exponentially m-isometric operators will be denoted by E_m. In [2] it has been proved that every m-isometry is an $(m+1)$-isometry and every invertible $2m$-isometry is a $(2m-1)$-isometry which implies that $E_{2m} = E_{2m-1}$.

Recall that $T \in B(H)$ is called an m-selfadjoint operator if

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k} T^{*k}T^{m-k} = 0$$

and T is skew-m-selfadjoint if iT is m-selfadjoint. These operators have been introduced and studied by Helton [5]. Moreover, for $m > 1$, the operator $T \in B(H)$ is said to be strict exponentially m-isometry if it is an exponentially m-isometric operator but not exponentially $(m-1)$-isometry. Similarly, one can define strict m-isometries and strict m-selfadjoint operators.

In [4, 6] authors investigate the sum of an m-isometric or an m-selfadjoint operator with a nilpotent operator and also the sum or product of two m-isometries or two m-selfadjoint operators. As an application of these results, we show that the sum of two commuting operators A and B that are, respectively, exponentially m-isometry and exponentially n-isometry is exponentially $(m+n-1)$-isometry. Also, we prove that if Q is a nilpotent operator of order l, $Q^l = 0$ and $Q^{l-1} \neq 0$, for some positive integer l, and A commutes with Q, then the sum $A + Q$ is an exponentially $(m + 2l - 2)$-isometric operator. It is known that the class of m-isometric and m-selfadjoint operators are stable under the powers [3, 6, 7]. We observe that the class of exponentially m-isometric operators is not stable under powers.

Also, we show that for each compact subset K of the pure imaginary line, there is an exponentially m-isometric operator T on a separable infinite Hilbert space whose spectrum is K. After that, we prove that limit of every sequence of exponentially m-isometric operators with respect to the strong operator topology is also an exponentially m-isometric operator. Furthermore, we show that every exponentially m-isometric diagonal, Toeplitz or multiplication operator is skew-m-selfadjoint. Moreover, we characterize normal, idempotent and weighted shift operators which are exponentially m-isometry.

2. MAIN RESULT

The skew-m-selfadjointness condition, $e^{-sT^*}e^{-sT} = \sum_{j=0}^{m-1} A_j s^j$ for each $s \in \mathbb{R}$ and some operators A_j, implies that the class of exponentially m-isometric operators contains all skew-m-selfadjoint operators. The following lemma implies that the class of skew-m-selfadjoint operators is a proper subset of E_m. In the following, $\langle .,. \rangle$ denotes the inner product on H. Moreover, for any vectors x and y in H, $x \otimes y$ denotes the rank one operator defined by

$$(x \otimes y)(z) = \langle z, y \rangle \, x.$$

Lemma 2.1. *Let $x, y \in H$. If $\langle x, y \rangle = 1$, then the following statements are equivalent:*
(a) $\|x\|\|y\| = 1$;
(b) *there exists a nonzero real number α such that $y = \alpha x$;*
(c) $\langle z, y \rangle \langle x, x \rangle y = \langle z, x \rangle \langle y, y \rangle x$, *for each $z \in H$;*
(d) $\langle x, z \rangle \langle z, y \rangle \geq 0$, *for each $z \in H$;*
(e) *$x \otimes y$ is selfadjoint.*

In the following example note that $x \otimes y$ is a nonzero idempotent if and only if $\langle x, y \rangle = 1$.

Example 2.2. Let H be an infinite-dimensional Hilbert space with an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$. For two distinct integers l and k greather that one, let $x = e_l$ and $y = e_l + e_k$. Then by Lemma 2.1, $x \otimes y$ is an idempotent which is not selfadjoint. Moreover, let A be the unilateral weighted shift operator, $Ae_j = w_j e_{j+1}$, with weight $(w_j)_j$ on H such that $w_l = w_{l-1} = w_{k-1} = 0$, $\prod_{i=0}^{N-1} w_{i+j} = 0$ for all j and $N = \left[\frac{m+1}{2}\right]$. Since A and iA are unitarily equivalent, Proposition 2.5 of [7] implies that A is a skew-m-selfadjoint operator. Also, it is easily seen that the operator $x \otimes y$ commutes with A. Thus $A + 2\pi i x \otimes y$ is an exponentially m-isometric operator that is not skew-m-selfadjoint.

It is known that m-isometric and m-selfadjont operators are stable under powers [3, 6, 7]; meanwhile exponentially m-isometric operators are not. As an example, the operator $(iI)^n$ is exponentially isometry for all odd numbers n but it is not for any even number m. The sum of the commuting exponentially m-isometries as follows.

Theorem 2.3. *Let $A, B, Q \in B(H)$ be commuting operators. Suppose that $A \in E_m$, $B \in E_n$ and $Q^l = 0$ for some positive integer l. Then*
(i) *For each $k \in \mathbb{Z}$, $kA \in E_m$.*
(ii) *$A + B \in E_{m+n-1}$. In particular, for every pure imaginary number μ, $A + \mu I \in E_m$.*
(iii) *$A + Q \in E_{m+2l-2}$.*

Moreover, $A + Q$ is strict exponentially $(m + 2l - 2)$-isometry if and only if $Q^{*l-1} \beta_{m-1}(e^A) Q^{l-1} \neq 0$. In particular, for the case $m = 1$, $A + Q$ is strict exponentially $(2l - 1)$-isometry if and only if Q is nilpotent of order l.

Now, similar description for m-isometric operators [4], we will describe exponentially m-isometric operators with prescribed spectrum.

Theorem 2.4. *Let H be an infinite dimensional separable Hilbert space and $m > 1$ be an odd number. If K is a non-empty compact subset of pure imaginary axis, then there exists a strict exponentially m-isometric operator $T \in B(H)$ with spectrum K.*

Proposition 2.5. *Let T be an exponentially m-isometric operator. If one of the following statements holds, then T is skew-selfadjoint.*
(i) *T is a Toeplitz operator.*
(ii) *T is a diagonal operator.*
(iii) *$T = M_\varphi$ is the multiplication operator defined by $M_\varphi f = \varphi f$ on $L_2(\mu)$, for a σ-finite measure μ and a bounded Borel function φ, or on the Hardy space H^2 for $\varphi \in H^\infty$.*

Proposition 2.6. *Let T be an exponentially m-isometric operator. Then the following statements hold:*
(i) *If T is a normal operator then it is exponentially isometry.*
(ii) *If T is bounded below then it is invertible. Consequently, if T is an isometric operator, then it is unitary.*
(iii) *If T is an idempotent operator then $T = 0$.*

Suppose that $(T_n)_{n\geq 1}$ is a sequence of operators in E_m. If T_n converges to T then $T \in E_m$. Now, we consider the following question: If T_n converges to T in the strong operator topology, is $T \in E_m$? We will give positive answer to this question.

Proposition 2.7. *If $(T_n)_{n\geq 1}$ is a sequence of operators in E_m that converges to T in the strong operator topology, then $T \in E_m$.*

Corollary 2.8. *Suppose that $(T_n)_{n\geq 1}$ is a sequence of m-selfadjoint operators such that $T_n \to T$ in the strong operator topology. Then T is also an m-selfadjoint operator.*

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Oral Presentation

FIXED POINT THEOREMS FOR θ-ϕ-CONTRACTION ON COMPLETE b-METRIC SPACES

PARASTOO HEIATIAN NAEINI

Department of Mmathematics, Payame Noor University , P. O. Box 19395-4697, Tehran, Iran
parastoo.heiatiannaeini@pnu.ac.ir

Abstract. In this paper, we introduce a new notion of generalized θ-ϕ-contraction and establish some results of fixed point for such mappings in complete b-metric space.

1. Introduction

The Banach contraction principle is a fundamental result in fixed point theory [2]. Due to its importance, various mathematics studied many interesting extensions and generalizations, (see [7]). In 2014, Jleli and Samet [5] analyzed a generalization of the Banach fixed point theorem on generalized metric spaces in a new type of contraction mappings called θ -contraction (or JS -contraction) and proved a fixed point result in generalized metric spaces. This direction has been studied and generalized in different spaces and various fixed point theorems have been developed (see [6]). Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, b -metric spaces were introduced by Bakhtin [1] and Czerwik [3], in such a way that triangle inequality is replaced by the b-triangle inequality: $d(x,y) \leq s(d(x,z)+d(z,y))$ for all pairwise distinct points x, y, z and $s \geq 1$. Any metric space is a b-metric space but in general, b-metric space might not be a metric space.

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Various fixed point results were established on such spaces. For more information on b -metric spaces and b-metric-like spaces, the readers can refer to (see [4]).

Very recently, Zheng et al. [8] introduced a new concept of θ-ϕ-contraction and established some fixed point results for such mappings in complete metric space and generalized the results of Brower and Kannan.

In this paper, we introduce a new notion of generalized θ-ϕ-contraction and establish some results of fixed point for such mappings in complete b-metric space. The results presented in the paper extend the corresponding results of Kannan and Reich on b-rectangular metric space.

Definition 1.1. Let X be a nonempty set, $s \geq 1$ be a given real number, and let $d : X \times X \to [0, +\infty[$ be a function such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

1. $d(x, y) = 0$, if only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$, ($b$-rectangular inequality).

Then (X, d) is called a b-metric space.

Definition 1.2. Let θ be the family of all functions $\theta :]0, +\infty[\to]1, +\infty[$such that

($\theta 1$) θ is increasing,
($\theta 2$) for each sequence $(x_n) \subset]0, +\infty[$;

$$lim_{n\to 0} x_n = 0 \ if \ and \ only \ if \ lim_{n\to\infty} \theta(x_n) = 1;$$

($\theta 3$) θ is continuous.

In [8], Zheng et al. presented the concept of θ-ϕ-contraction on metric spaces and proved the following nice result.

Definition 1.3. Let ϕ be the family of all functions $\phi : [1, +\infty[\to [1, +\infty[$, such that

($\varphi 1$) ϕ is nondecreasing;
($\varphi 2$) for each $t \in]1, +\infty[$, $lim_{n\to\infty} \phi^n(t) = 1$;
($\varphi 3$) ϕ is continuous.

Definition 1.4. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be a θ-ϕ-contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \theta[d(Tx, Ty)] \leq \phi(\theta[N(x, y)]),$$

where

$$N(x, y) = max\{(x, y), d(x, Tx), d(y, Ty)\}.$$

Theorem 1.5. *Let (X, d) be a complete metric space and let $T : X \to X$ be a θ-ϕ-contraction. Then T has a unique fixed point.*

2. MAIN RESULTS

In this paper, using the idea introduced by Zheng et al., we present the concept θ-ϕ-contraction in b-metric spaces and we prove some fixed point results for such spaces.

Definition 2.1. Let (X, d) be a b-metric space with parameter $s > 1$ space and $T : X \to X$ be a mapping.

(1) T is said to be a θ-contraction if there exist $\theta \in \Theta$ and $r \in]0, 1[$ such that

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^3 d(Tx, Ty)] \leq \theta[M(x, y)]^r,$$

where

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s^2}\}.$$

(2) T is said to be a θ-ϕ-contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^3 d(Tx, Ty) \leq \phi[\theta(M(x, y))],$$

where

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2s^2}\}.$$

(3) T is said to be a θ-ϕ- Kannan-type contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$, we have

$$\theta[s^3 d(Tx, Ty)) \leq \phi[\theta(\frac{d(x, Tx) + d(y, Ty)}{2})].$$

(4) T is said to be a θ-ϕ-Reich-type contraction if there exist exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$, we have

$$\theta[s^3 d(Tx, Ty)) \leq \phi[\theta(\frac{d(x, y) + d(x, Tx) + d(y, Ty)}{3})].$$

Theorem 2.2. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a θ-contraction, i.e, there exist $\theta \in \Theta$ and $r \in]0, 1[$ such that for any $x, y \in X$, we have*

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^3 d(Tx, Ty)] \leq \theta[M(x, y)]^r.$$

Then T has a unique fixed point.

Corollary 2.3. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\theta \in \Theta$and $k \in]0, 1[$ such that for any $x, y \in X$, we have*

$$d(Tx, Ty) > 0 \Rightarrow \theta[s^3 d(Tx, Ty)] \leq [\theta(d(x, y))]^k.$$

Theorem 2.4. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y \in X$,*

$$d(Tx,Ty) > 0 \Rightarrow \theta[s^3 d(Tx,Ty)] \leq \phi[\theta(M(x,y))]$$

where

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2s^2}d(y,Tx)\}.$$

Then T has a unique fixed point.

It follows from Theorem 2.4 that we obtain the followed fixed point theorems for θ-ϕ-Kannan-type contraction and θ-ϕ-Reich-type contraction. The results presented in the paper improve and extend the corresponding results due to Kannan-type contraction and Reich-type contraction on rectangular b-metric space.

Theorem 2.5. *Let (X,d) be a complete b-metric space and $T : X \to X$ be a Kannan-type contraction. Then T has a unique fixed point.*

Theorem 2.6. *Let (X,d) be a complete b-metric space and $T : X \to X$ be a Reich-type contraction. Then T has a unique fixed point.*

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Oral Presentation

EXISTENCE AND UNIQUENESS OF A FIXED POINT THEOREM ON C_J- METRIC SPACES

PARASTOO HEIATIAN NAEINI

Department of Mmathematics, Payame Noor University , P. O. Box 19395-4697, Tehran, Iran
parastoo.heiatiannaeini@pnu.ac.ir

Abstract. In this paper, we generalize a $J-$ metric spaces, where it defined a metric space in three dimensions with a triangle inequality that includes a constant $b > 0$. We extend the notion of $J-$ metric spaces to C_J metric spaces that include a control function θ in three dimensions instead of the constant b.

1. Introduction

The fixed point theory is a new, essential theory, and its application is utilized in many fields, including Mathematics, Economics, and many others. For example, the impact of the fixed point theory in the fractional differential equations appear clearly to all the observers, see [3, 4]. The fixed point theory and the proof of the uniqueness were introduced by Banach [2], which was encouraging to all subsequent researchers to start working on this theory; see [5, 8].

These days, the fixed point is an active area wildly generalizing Banach, see [6, 7]. Generalization of the fixed point theory can be made in two ways, either a generalization of the Banach contraction to another linear or nonlinear contraction. The other way of extension is to generalize the metric spaces by either changing the triangle inequality, omitting the symmetry condition, or assuming that the self-distance is not necessarily zero.

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Those generalizations are important due to the fact that more general spaces or contractions impact a greater number of applications that can be adapted to that results.

In this work, we introduce a new extension to $J-$ metric spaces, called C_J- metric spaces, where θ is the controlled function in the triangle inequality. We prove some fixed point results in this new type of metric space.

In the main result section, we prove the existence and the uniqueness of a fixed point for self-mappings on C_J- metric spaces, in Theorems 2.7 and 2.8, we consider self-mappings that satisfy linear contractions where in Theorem 2.9, we consider mappings that satisfy nonlinear contractions. Our finding generalizes many results in the literature.

We begin our preliminaries by recalling the definitions of J-metric spaces.

Definition 1.1. [1] Consider a nonempty set δ, and a function $J : \delta^3 \to [0, \infty)$. Let us define the set,

$$S(J, \delta, \phi) = \{\{\phi_n\} \subset \delta : lim_{n\to\infty} J(\phi, \phi, \phi_n) = 0\}$$

for all $\phi \in \delta$.

Definition 1.2. [1] Let δ be a set with at least one element and, $J : \delta^3 \to [0, \infty)$ that satisfies the mentioned below conditions:

(i) $J(\alpha, \beta, \gamma) = 0$ implies $\alpha = \beta = \gamma$ for any $\alpha, \beta, \gamma \in \delta$.

(ii) There are some $b > 0$, where for each $(\alpha, \beta, \gamma) \in \delta^3$ and $\{\nu_n\} \in S(J, \delta, \nu)$

$$J(\alpha, \beta, \gamma) \leq b\ limsup_{n\to\infty}(J(\alpha, \alpha, \nu_n) + J(\beta, \beta, \nu_n) + J(\gamma, \gamma, \nu_n)).$$

Then, (δ, J) is defined as a $J-$metric space. In addition, if $J(\alpha, \alpha, \beta) = J(\beta, \beta, \alpha)$ for each $\alpha, \beta \in \delta$, the pair (δ, J) is defined as a symmetric $J-$metric space.

2. MAIN RESULTS

In this paper, we will define C_J- metric spaces and prove the existence and the uniqueness of the fixed point of self-mapping.

Definition 2.1. Let δ is a non empty set and a function $C_J : \delta^3 \to [0, \infty)$. Then the set is defined as follows

$$S(C_J, \delta, \alpha) = \{\{\alpha_n)\} \subset \delta : lim_{n\to\infty} C_J(\alpha, \alpha, \alpha_n) = 0\}$$

for each $\alpha \in \delta$.

Definition 2.2. Let δ be a set with at least one element and $C_J : \delta^3 \to [0, \infty)$ fulfill the following conditions:

(i) $C_J(\alpha, \beta, \gamma) = 0$ implies $\alpha = \beta = \gamma$ for any $\alpha, \beta, \gamma \in \delta$.

(ii) (ii) There exist a function $\theta : \delta^3 \to [0, \infty)$, where θ is a continuous function and

$$lim_{n\to\infty} \theta(\alpha, \alpha, \alpha_n)$$

is a finite and exist where,

$C_J(\alpha,\beta,\gamma) \leq \theta(\alpha,\beta,\gamma) limsup_{n\to\infty}(C_J(\alpha,\alpha,\phi_n)+C_J(\beta,\beta,\phi_n)+C_J(\gamma,\gamma,\phi_n)$.

Then (δ, C_J) is defined as C_J-metric space. In addition, if

$$C_J(\alpha,\alpha,\beta) = C_J(\beta,\beta,\alpha)$$

for each $\alpha,\beta \in \delta$, then (δ, C_J) is defined as symmetric C_J-metric space.

Remark 2.3. Notice that, this symmetry hypothesis does not necessarily mean that

$$C_J(\alpha,\beta,\gamma) = C_J(\beta,\alpha,\gamma) = C_J(\gamma,\beta,\alpha) = \cdots .$$

We will start by presenting some properties in the topology of C_J-metric spaces.

Definition 2.4. (1) Let (δ, C_J) is a C_J- metric space. A sequence $\{\alpha_n\} \subset \delta$ is convergent to an element $\alpha \in \delta$ if $lim_{n\to\infty}\alpha_n = \alpha$,for $\{\alpha_n\} \in S(C_J,\delta,\alpha)$.

(2) Let (δ, C_J) is a C_J-metric space. A sequence $\{\alpha_n\} \subset \delta$ is called Cauchy iff $lim_{n,m\to\infty}C_J(\alpha_n,\alpha_n,\alpha_m) = 0$.

(3) A C_J- metric space is called complete if each Cauchy sequence in δ is convergent.

(4) In a C_J-metric space (α, C_J), if ψ is a continuous map at $\alpha_0 \in \Gamma\delta$ then for each $\alpha_n \in S(C_J,\alpha,a_0)$ gives $\{\psi a_n\} \in S(C_J,\alpha,\psi a_0)$.

Proposition 2.5. *In a C_J-metric space (δ, C_J), if $\{\alpha_n\}$ converges, then it is convergent to one exact element in δ.*

Definition 2.6. Let (δ, C_{J_1}) and (Γ, C_{J_1}) are two C_J- metric spaces and $\psi : \delta \to \Gamma$ is a map . Then ψ is said to be a continuous at $a_0 \in \delta$ if, for each $\varepsilon > 0$, there is $\zeta > 0$ where, for each $a \in \delta$, $C_{J_2}(\psi a_0, \psi a_0, \psi\alpha) < \varepsilon$ whenever $C_{J_1}(a_0, a_0, \alpha) < \zeta$.

Theorem 2.7. *Let (δ, C_J) is a C_J- complete symmetric metric space, and $g : \delta \to \delta$ is a continuous map satisfies*

$$C_J(g\alpha, g\beta, g\gamma) \lneq P(C_J(\alpha,\beta,\gamma)) \ forall\ \alpha,\beta,\gamma \in \delta.$$

Where, $P : [0,+\infty) \to [0,+\infty)$ is a function and for all $t \in [0,+\infty)$,

$$t \succ x, P(t) \succ P(x).$$

$$lim_{n\to\infty}P^n(t) = 0 \ for\ each\ fixed\ t > 0.$$

Then, g has a unique fixed point in δ.

Theorem 2.8. *Let (δ, C_J) is a C_J- complete symmetric metric space and $g : \delta \to \delta$ be a mapping that satisfies,*

$$C_J(g\alpha, g\beta, g\gamma) \leq \phi(\alpha,\beta,\gamma)C_J(\alpha,\beta,\gamma),\ \forall\alpha,\beta,\gamma \in \delta,$$

where $\phi \in A$, and $\phi : \delta^3 \to (0,1)$, such that

$$\phi(g(\alpha,\beta,\gamma)) \leq \phi(\alpha,\beta,\gamma) \ and\ \{g : \delta \to \delta\}$$

g is a given mapping. Then g has a unique fixed point in δ.

Theorem 2.9. *Let (δ, C_J) is a complete symmetric C_J-metric spaces, $g : \delta \to \delta$ is a continuous map where*

$$C_J(g\alpha, g\beta, g\gamma) \leq aC_J(\alpha, \beta, \gamma)+bC_J(\alpha, g\alpha, g\alpha)+cC_J(\beta, g\beta, g\beta)+dC_J(\gamma, g\gamma, g\gamma)$$

for each $\alpha, \beta, \gamma \in \delta$ where

$$0 < a + b < 1 - c - d,$$
$$0 < a < 1.$$

Then, there is a unique fixed point of g.

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Oral Presentation $\qquad$ $*$: Speaker

ORTHOGONALLY GENERALIZED JENSEN-TYPE $\rho-$FUNCTIONAL EQUATION

MOHAMMAD TAGHI HEYDARI* AND VAHID KESHAVARZ

Department of Mathematics, College of Sciences, Yasouj University, Yasouj, 75918 Iran
heydari@yu.ac.ir and v.keshavarz68@yahoo.com

Abstract. In this paper, we introduce and solve the concept of the generalized Jensen-type $\rho-$functional equation. Finally, we investigate the Hyers-Ulam stability of generalized Jensen-type $\rho-$functional equation with Găvruta's control function on orthogonally Banach algebras approach direct methods.

1. Introduction

A classical question in the sense of functional equation says that "when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation? " Ulam raised the stability of functional equations and Hyers was the first one which gave an affirmative answer to the question of Ulam for additive mapping between Banach spaces. Th. M. Rassias proved a generalized version of the Hyers's theorem for approximately additive maps. Găvruta generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference general control function $\varphi(x, y)$. The study of stability problem of functional equations have been done by several authors on different spaces such as Banach, C^*-Banach algebras and modular spaces (for example see [2, 3, 4, 6]).

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* Speaker.

Recently, Eshaghi Gordji et al. [1] introduced notion of the orthogonal. The study on orthogonal sets has been done by several authors (for example, see [5, 7, 8])

Definition 1.1. [1] Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be a binary relation. If there exists $u_0 \in X$ such that for all $v \in X$,

$$v \perp u_0 \;\; or \;\; u_0 \perp v,$$

then $\perp$ is called an orthogonally set (briefly O-set). We denote this O-set by $(X, \perp)$. Let $(X, \perp)$ be an O-set and (X, d) be a generalized metric space, then $(X, \perp, d)$ is called orthogonally generalized metric space.

Let $(X, \perp, d)$ be an orthogonally metric space.
(i) A sequence $\{u_n\}_{n\in\mathbb{N}}$ is called orthogonally sequence (briefly O-sequence) if for any $n \in \mathbb{N}$,

$$u_n \perp u_{n+1} \;\; or \;\; u_{n+1} \perp u_n.$$

(ii) Mapping $f : X \to X$ is called $\perp-$continuous in $u \in X$ if for each O-sequence $\{u_n\}_{n\in\mathbb{N}}$ in X with $u_n \to u$, then $f(u_n) \to f(u)$. Clearly, every continuous map is $\perp-$continuous at any $u \in X$.
(iii) $(X, \perp, d)$ is called orthogonally complete (briefly O-complete) if every Cauchy O-sequence is convergent to a point in X.
(iv) Mapping $f : X \to X$ is called $\perp$-preserving if for all $u, v \in X$ with $u \perp v$, then $f(u) \perp f(v)$.
Consider the orthogonally generalized Jensen-type ρ-functional equation

$$\begin{aligned}
&f\big(\frac{u+v}{2}+w\big)+f\big(\frac{u+w}{2}+v\big)+f\big(\frac{v+w}{2}+u\big)-2f(u)-2f(v)-2f(w)=\\
&\rho\Big(3f\big(\frac{u+v+w}{3}\big)-2f\big(\frac{u+v}{2}\big)-2f\big(\frac{u+w}{2}\big)-2f\big(\frac{v+w}{2}\big)\\
&+f(u)+f(v)+f(w)\Big)
\end{aligned} \tag{1.1}$$

such that $\rho \neq 0, \pm 1$ is a real number and $u \perp v,\ u \perp w,\ v \perp w$.
In this paper, we investigate (1.1) is additive equation and the Hyers-Ulam of it's equation approach direct methods with Găvruta's control function.

2. Hyers-Ulam Stability

Throughout this section, let A and B are two orthogonally Banach algebras.
To prove the main theorem, we need the following lemma. Firstly, in the next lemma, we prove that f is an additive mapping.

Lemma 2.1. *If a mapping $f : A \to B$ satisfies*

$$f\big(\frac{u+v}{2}+w\big)+f\big(\frac{u+w}{2}+v\big)+f\big(\frac{v+w}{2}+u\big)-2f(u)-2f(v)-2f(w)= \rho\Big(3f\big(\frac{u+v+w}{3}\big)-2f\big(\frac{u+v}{2}\big)-2f\big(\frac{u+w}{2}\big)-2f\big(\frac{v+w}{2}\big) + f(u)+f(v)+f(w)\Big) \tag{2.1}$$

for all $u, v, w \in A$ with $u\perp v,\ u\perp w,\ v\perp w$, then the mapping f is additive.

In the following theorem, we prove Hyers-Ulam stability of orthogonally generalized Jensen-type ρ-functional with Găvruta's control function on orthogonally Banach algebras.

Theorem 2.2. *Let $\varphi : A^3 \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(u, v, w) := \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi(2^n u, 2^n v, 2^n w) < \infty \tag{2.2}$$

Suppose that $f : A \to B$ is a mapping satisfying

$$\Big\|f\big(\frac{u+v}{2}+w\big)+f\big(\frac{u+w}{2}+v\big)+f\big(\frac{v+w}{2}+u\big)-2f(u)-2f(v)-2f(w)- \rho\Big(3f\big(\frac{u+v+w}{3}\big)-2f\big(\frac{u+v}{2}\big)-2f\big(\frac{u+w}{2}\big)-2f\big(\frac{v+w}{2}\big) + f(u)+f(v)+f(w)\Big)\Big\| \le \varphi(u, v, w) \tag{2.3}$$

for all $u, v, w \in A$ with $u\perp v,\ u\perp w,\ v\perp w$. Then there exist a unique additive mapping $T : A \to B$ such that

$$\|f(u) - T(u)\| \le \frac{1}{3}\varphi(u, u, u)$$

for all $u \in A$.

In the next corollary we prove the Hyers-Ulam stability of orthogonally generalized Jensen-type ρ-functional with Rassias's control function on orthogonally Banach algebras.

Corollary 2.3. *Let $\theta, p_i, q_i,\ i = 1, 2, 3$ are positive real such that $p_i < 1$ and $q_i < 3$. Suppose that $f : A \to B$ is a mapping such that*

$$\Big\|f\big(\frac{u+v}{2}+w\big)+f\big(\frac{u+w}{2}+v\big)+f\big(\frac{v+w}{2}+u\big)-2f(u)-2f(v)-2f(w)- \rho\Big(3(f\big(\frac{u+v+w}{3}\big)-2f\big(\frac{u+v}{2}\big)-2f\big(\frac{u+w}{2}\big)-2f\big(\frac{v+w}{2}\big) + f(u)+f(v)+f(w)\Big)\Big\| \le \theta(\|u\|^{p_1}+\|v\|^{p_2}+\|w\|^{p_3}), \tag{2.4}$$

for all $u, v, w \in A$ *with* $u \perp v,\ u \perp w,\ v \perp w$. *Then there exist a unique additive mapping* $T : A \to B$ *such that*

$$\|f(u) - T(u)\| \leq \frac{\theta}{3}\{\frac{1}{2-2^{p_1}}\|u\|^{p_1} + \frac{1}{2-2^{p_2}}\|u\|^{p_2} + \frac{1}{2-2^{p_3}}\|u\|^{p_3}\}$$

for all $u \in A$.

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Oral Presentation

A CONDITIONAL OPERATOR ON C^*-ALGEBRAS

M. R. JABBARZADEH

Faculty of Mathematical Sciences, University of Tabriz, P. O. Box: 51666-15648, Tabriz, Iran
mjabbar@tabrizu.ac.ir

Abstract. In this note, we introduce a lower triangular conditional operator on a unital C^*-algebra $\mathcal{A}$.

1. Introduction

A linear mapping $E : \mathcal{A} \to \mathcal{B}$ is called a projection if $E(b) = b$ for every $b \in \mathcal{B}$. In this case $E^2 = E$ and $\|E\| \geq 1$. Tomiyama in [8] prove that if E is a projection of norm 1 from $\mathcal{A}$ onto $\mathcal{B}$, then E is positive, $E(a^*)E(a) \leq E(a^*a)$ and $\mathcal{B}$-linear, that is, $E(b_1ab_2) = b_1E(a)b_2$ for all $a \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$. A $\mathcal{B}$-linear projection $E : \mathcal{A} \to \mathcal{B}$ which is also a positive mapping, is called a conditional expectation([1, 2, 4, 5, 6, 7, 8]).

Let $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. We denote by L_a the left multiplication operator on $\mathcal{A}$. Define the linear operator $T_a : \mathcal{A} \to \mathcal{A}$ by $T_a(x) = E(a)x + aE(x) - E(a)E(x)$, where $E : \mathcal{A} \to \mathcal{B}$ is a conditional expectation operator. Each $a \in \mathcal{A}$ can be written uniquely as $a = a_1 + a_2$ where $a_1 = E(a) \in \mathcal{B}$ and $a_2 = a - E(a) \in \mathcal{N}(E)$, because $\mathcal{A} = \mathcal{B} \oplus \mathcal{N}(E)$. It follows that $T_a = L_{a_1} + L_aE - L_{a_1}E = L_{a_1} + L_{a_2}E$. Thus, $\alpha T_a + T_b = T_{\alpha a+b}$, $T_a(\mathcal{N}(E)) \subseteq \mathcal{N}(E)$ and $\|T_a\| \leq 3\|a\|$. When $e = 1$ then $T_1 = I$, the identity operator. The matrix representation of T_a with respect to the decomposition $\mathcal{A} = \mathcal{B} \oplus \mathcal{N}(E)$

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is

$$T_a = \begin{bmatrix} L_{a_1} & 0 \\ L_{a_2} & L_{a_1} \end{bmatrix},$$

where $a = a_1 + a_2$. Put $a \star b = a \star_E b = T_a(b)$. Then $a \star b = a_1 b + ab_1 - a_1 b_1 = a_1 b_1 + (a_1 b_2 + a_2 b_1)$. So $(a \star b)_1 = a_1 b_1$ and $(a \star b)_2 = a_1 b_2 + a_2 b_1$. It follows that

$$T_a T_b = \begin{bmatrix} L_{a_1} & 0 \\ L_{a_2} & L_{a_1} \end{bmatrix} \begin{bmatrix} L_{b_1} & 0 \\ L_{b_2} & L_{b_1} \end{bmatrix} = \begin{bmatrix} L_{(a\star b)_1} & 0 \\ L_{(a\star b)_2} & L_{(a\star b)_1} \end{bmatrix} = T_{a\star b}.$$

Put $\mathcal{K} = \mathcal{K}(E) = \{T_a = L_{a_1} + L_{a_2} E : a \in \mathcal{A}\}$. Then $\mathcal{K}$ is a subalgebra of $B(\mathcal{A})$, the Banach algebra of all bounded and linear maps defined on $\mathcal{A}$ and with values in $\mathcal{A}$. Note that the mapping $\mathcal{T} : \mathcal{A} \to \mathcal{K}$ given by $\mathcal{T}(a) = T_a$ is linear with $\|\mathcal{T}\| \leq 3$ and $\mathcal{T}(a \star b) = \mathcal{T}(a)\mathcal{T}(b)$ for all $a, b \in \mathcal{A}$.

2. Characterizations

Let E_1, E_2 be two distinct conditional expectations from $\mathcal{A}$ onto $\mathcal{B}$. Then it is easy to check that $G := E_1 + E_2 - I$ is invertible. Since $E_1 E_2 = E_2$ and $E_2 E_1 = E_1$, then we have $E_1 G = E_2 = G E_2$, $E_2 G = E_1 = G E_1$ and $(I - E_2)(I - E_1) = I - E_2$.

Proposition 2.1. *For $a \in \mathcal{A}$, let $T_a \in \mathcal{K}(E_1)$ and $S_a \in \mathcal{K}(E_2)$. Then there is an invertible operator G on $\mathcal{A}$ such that $G T_a G = S_{G(a)}$ and the mapping $\Lambda : T_a \to G T_a G$ is an algebra isomorphism of $\mathcal{K}(E_1)$ onto $\mathcal{K}(E_2)$ which is a homeomorphism.*

Proof. Take $G = E_1 + E_2 - I$. Then G is invertible with $G^{-1} = G$. Recall that for each $a, b \in \mathcal{A}$, $T_a(b) = a \star_{E_1} b = (E_1 a) b + a(E_1 b) - (E_1 a)(E_1 b)$ and $S_a(b) = a \star_{E_2} b = (E_2 a) b + a(E_2 b) - (E_2 a)(E_2 b)$. Then we have

$$\begin{aligned} (T_a G)(b) &= T_a(E_1 b + E_2 b - b) \\ &= a \star_{E_1} (E_1 b) + a \star_{E_1} (E_2 b) - a \star_{E_1} b \\ &= a(E_2 b) + (E_1 a)(E_1 b) - (E_1 a) b, \end{aligned}$$

and

$$\begin{aligned} (G T_a G)(b) &= (E_1 + E_2 - I)[a(E_2 b) + (E_1 a)(E_1 b) - (E_1 a) b] \\ &= E_2 b - a(E_2 b) + (E_1 a) b. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} S_{Ga}(b) &= (Ga) \star_{E_2} b = E_2(Ga) b + (Ga)(E_2 b) - E_2(Ga) E_2(b) \\ &= (E_1 a) b + (E_2 a)(E_2 b) - a(E_2 b). \end{aligned}$$

Thus, $G T_a G = S_{Ga} \in \mathcal{K}(E_2)$. Also, $\Lambda(T_a T_b) = \Lambda(T_a)\Lambda(T_b)$. So, Λ is a continuous algebra isomorphism and $\Lambda^{-1}(S_b) = G S_b G$ is also continuous with respect to any of the operator topologies. $\square$

Proposition 2.2. *Let $a \in \mathcal{A}$. If a_1 has a left inverse, then T_a is injective. Moreover, if $\mathcal{B}$ has a right invertible element, then the mapping $\mathcal{T} : \mathcal{A} \to \mathcal{K}$ given by $\mathcal{T}(a) = T_a$ is injective.*

Proof. Let $T_a(b) = 0$ for some $b \in \mathcal{A}$. Then $a_1b_1 = -(a_1b_2 + a_2b_1) \in \mathcal{A} \cap \mathcal{N}(E) = \{0\}$ and so $b_1 = 0$. It follows that $a_1b = 0$ and hence $b = 0$. Now, let $b_0 \in \mathcal{B}$ is a right invertible element and let $T_a(b) = a_1b + ab_1 - a_1b_1 = 0$ for all $b \in \mathcal{A}$. Take $b = e$. Then $ae = 0$ and so $a_1 = E(ae) = 0$. Thus, $a_2b_1 = 0$ for all $b_1 \in \mathcal{B}$. Take $b_1 = b_0$. Then $a_2 = 0$. Consequently, $a = 0$. □

Proposition 2.3. *Let $S_0(\mathcal{A}|\mathcal{B}) = \{x \in \mathcal{A} : \mathcal{A}ex \subseteq \mathcal{B}\}$. Then the following assertions hold.*

(i) $\mathcal{N}e + e\mathcal{N} + \mathcal{B} = \vee_{a\in A}\mathcal{R}(T_a)$, where $\vee$ denotes the algebraic span.

(ii) $\cup_{a\in A}T_a(S_0) \subseteq \mathcal{B}$.

(iii) $\mathcal{N} \subseteq \cap_{a\in\mathcal{N}}\mathcal{N}(T_a)$. Moreover, if $\mathcal{N}$ has a left invertible element, then $\mathcal{N} = \cap_{a\in\mathcal{N}}\mathcal{N}(T_a)$.

Proof. (i) Let $a, x \in A$. Then $T_a(x) = (a_2x_1)e + e(a_1x_2) + a_1x_1 \in \mathcal{N}e + e\mathcal{N} + \mathcal{B}$ and hence $\vee_{a\in A}\mathcal{R}(T_a) \subseteq \mathcal{N}e + e\mathcal{N} + \mathcal{B}$. Conversely, let $k \in \mathcal{N}$ and $b \in \mathcal{B}$. Since $ek = T_1(k)$, $ke = T_k(1)$ and $b = T_b(e)$, then $\mathcal{N}e + e\mathcal{N} + \mathcal{B} \subseteq \vee_{a\in A}\mathcal{R}(T_a)$.

(ii) Let $a \in A$ and $x \in S_0$. Then $\{ex, aex\} \subset \mathcal{B}$, $x_1 = E(x) = E(ex) = ex$ and so $T_a(x) = a_1x + ax_1 - a_1x_1 = a_1ex + aex - a_1ex = aex \in \mathcal{B}$. Thus, $\cup_{a\in\mathcal{A}}T_a(S_0) \subseteq \mathcal{B}$.

(iii) Let $\{a, x\} \subset \mathcal{N}$. Then $a_1 = 0 = x_1$, $T_a(x) = 0$ and so $x \in \mathcal{N}(T_a)$ for all $a \in \mathcal{N}$. Now let $x \in \cap_{a\in\mathcal{N}}\mathcal{N}(T_a)$ and for some $a_2 \in \mathcal{N}$, there is an element $a_0 \in \mathcal{A}$ such that $a_0a_2 = 1$. Then $a_2x_1 = T_{a_2}(x) = 0$ and hence $x_1 = a_0a_2x_1 = 0$. Thus, $x = x_2 \in \mathcal{N}$. □

Proposition 2.4. *Let $a, b \in A$. Then the equation $T_aX = T_b$ has a solution in $\mathcal{K}$ whenever a_1 has a left inverse.*

Proof. Let $a_0a_1 = 1$ and $X = T_x$ for some $a_0, x \in \mathcal{A}$. According to the matrix form of $T_aT_x = T_b$, we have

$$\begin{bmatrix} L_{a_1x_1} & 0 \\ L_{a_2x_1+a_1x_2} & L_{a_1x_1} \end{bmatrix} = \begin{bmatrix} L_{b_1} & 0 \\ L_{b_2} & L_{b_1} \end{bmatrix}.$$

It follows that $a_1x_1 = b_1$ and $a_2x_1 + a_1x_2 = b_2$. Thus, $x_1 = a_0b_1$ and $a_1x_2 = b_2 - a_2x_1$. Then $x_2 = a_0b_2 - a_0a_2a_0b_1$ and hence $x = x_1 + x_2 = a_0b + a_0a_2a_0b_1$. □

It has been shown in [3, Lemma 1.4, Proposition 3.1] that $s\mathcal{N} = \mathcal{N}s = 0$ and $\|b + S_1\|_{\frac{\mathcal{B}}{S_1}} = \|L_b\|_{\mathcal{N}\to\mathcal{N}}$, for all $s \in S_1$ and $b \in \mathcal{B}$. Using these, we have the following result.

Proposition 2.5. *Let $\|I - E\| \leq 1$. Then*

$$\|L_{a_1}\|_{\mathcal{N}\to\mathcal{N}} \leq \inf_{k\in\mathcal{N}} \|T_{a+k}\| \leq \|T_{a_1}\| \leq \|L_{a_1}\|_{\mathcal{B}\to\mathcal{B}} + \|L_{a_1}\|_{\mathcal{N}\to\mathcal{N}}.$$

Proof. Let $s \in S_1$ and $x \in \mathcal{A}$ with $\|x\| = 1$. Since E is a contraction, then we have $\|x_1\| = \|E(x)\| \leq \|E\| \, \|x\| \leq \|x\| = 1$ and $\|x_2\| = \|(I-E)x\| \leq \|I-E\| \, \|x\| \leq 1$. Then we get that

$$\begin{aligned}\|T_{a_1}x\| &= \|a_1x_1 + a_1x_2\| \leq \|a_1x_1\| + \|a_1x_2\| \\ &= \|a_1x_1\| + \|(a_1+s)x_2\| \leq \sup_{x_1 \in B} \|a_1x_1\| + \|a_1+s\| \\ &= \|L_{a_1}\|_{B\to B} + \inf \|a_1+s\| = \|L_{a_1}\|_{B\to B} + \|a_1+S_1\|_{\frac{B}{s_1}}.\end{aligned}$$

Thus, $\|T_{a_1}\| \leq \|L_{a_1}\|_{B\to B} + \|L_{a_1}\|_{\mathcal{N}\to\mathcal{N}}$. Also, we have

$$\inf_{k\in\mathcal{N}} \|T_{a+k}\| \leq \|T_{a+(a_1-a)}\| = \|T_{a_1}\|.$$

On the other hand, $\|T_{a+k}\| = \|T_{a_1+(a_2+k)}\| = \sup_{\|x\|=1} \|a_1x + (a_2+k)x_1\| \geq \sup_{\|x\|=1} \|a_1x_2\| = \|L_{a_1}\|_{\mathcal{N}\to\mathcal{N}}$. Hence, $\inf_{k\in\mathcal{N}} \|T_{a+k}\| \geq \|L_{a_1}\|_{\mathcal{N}\to\mathcal{N}}$. □

Proposition 2.6. *$\mathcal{K}$ is closed in the norm operator topology.*

Proof. Let $\{T_{a_n}\} \subseteq \mathcal{K}$ and $\|T_{a_n} - T\| \to 0$, for some $T \in B(\mathcal{A})$. Then we have

$$\lim_{n\to\infty} T_{a_n} = \lim_{n\to\infty} \begin{bmatrix} L_{a_{n1}} & 0 \\ L_{a_{n2}} & L_{a_{n1}} \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = T$$

where $a_{n1} = E(a_n)$ and $a_{n2} = a_n - E(a_n)$. Since $T_{a_n}(\mathcal{N}) \subseteq \mathcal{N}$ then $T(\mathcal{N}) \subseteq \mathcal{N}$, and so $T_2 = 0$. Further, $\lim_{n\to\infty} \|T_{a_{n1}} - T_1\|_{\mathcal{B}\to\mathcal{B}} = 0$ implies that $\lim_{n\to\infty} a_{n1} = \lim_{n\to\infty} a_{n1}e = T_1e = ETEe = E(Te)$, and so $T_1x_1 = \lim_{n\to\infty} a_{n1}x_1 = E(Te)x_1$ for all $x_1 \in \mathcal{B}$. Thus, $T_1 = L_{E(Te)}$. Likewise, for each $x_2 \in \mathcal{N}$ we have $T_4x_2 = \lim_{n\to\infty} a_{n1}x_2 = E(Te)x_2$ and hence $T_4 = L_{E(Te)}$. Moreover, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} T_{a_n}1 = T1$, then for each $x_1 \in \mathcal{B}$ we obtain that $T_3x_1 = \lim_{n\to\infty} a_{n2}x_1 = \lim_{n\to\infty}(a_n - a_{n1})x_1 = (T1 - E(Te))x_1$. Cosequently, $T1 - E(Te) \in \mathcal{N}$, $T_3 = L_{T1-E(Te)}$ and

$$T = \begin{bmatrix} L_{E(Te)} & 0 \\ L_{T1-E(Te)} & L_{E(Te)} \end{bmatrix} \in \mathcal{K}.$$

□

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Oral Presentation **$*$: Speaker**

HOMOCLINIC ORBIT IN A LIÉNARD TYPE SYSTEM

TOHID KASBI GHARAHASANLOU* AND VAHID ROOMI

Department of mathematics and statistics, Faculty and Research Institute of Basic Sciences, Imam Hossein University, Tehran, Iran
Department of mathematics, Faculty of Basic Sciences, Azarbaijan Shahid Madani University, East Azearbaijan, Iran
t.kasbi@ihu.ac.ir; roomi@azaruniv.ac.ir

Abstract. The object of this paper is to study the existence of an important orbit in a generalized Liénard type system. This trajectory is doubly asymptotic to an equilibrium solution, i.e., an orbit which lies in the intersection of the stable and unstable manifolds of a critical point. Such an orbit is called a homoclinic orbit.

1. Introduction

Consider the planar system

$$\begin{aligned} \dot{x} &= P(Q(y) - F(x)) \\ \dot{y} &= -g(x), \end{aligned} \tag{1.1}$$

which is a generalized Liénard type system, where P, Q, F and g are continuous functions satisfying suitable assumptions in order to ensure the existence of a unique solution to the initial value problems. Moreover, suppose that the following assumptions hold under which the origin is the unique critical point of system (1.1).

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* Speaker.

$P(u)$ and $Q(y)$ are strictly increasing and $F(0) = P(0) = Q(0) = 0$, $uP(u) > 0$ for $u \neq 0$, $yQ(y) > 0$ for $y \neq 0$ and $xg(x) > 0$ for $x \neq 0$.

System (1.1) includes the classical Liénard system as a special case, which is of great importance in various applications.

Definition 1.1. In system (1.1), a trajectory is said to be a homoclinic orbit if its $\alpha-$ and $\omega-$limit sets are the origin (see Fig. 1.1).

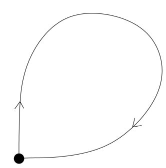

FIGURE 1. Homoclinic Orbit

The main purpose of this paper is to give an implicit necessary and sufficient condition and some explicit sufficient conditions on $F(x)$, $g(x)$, $P(u)$ and $Q(y)$ under which system (1.1) has homoclinic orbits. These results extend and improve the results presented for special cases of system (1.1) in [1, 3, 4].

The curve $\Gamma = \{(x, y)|y = Q^{-1}(F(x))\}$ is called the characteristic curve of (1.1). Let

$$\Gamma_1 = \{(x, y)|y = Q^{-1}(F(x)) \ \ and \ \ x > 0\},$$

and

$$\Gamma_2 = \{(x, y)|y = Q^{-1}(F(x)) \ \ and \ \ x < 0\}.$$

Then, $\Gamma = \Gamma_1 \bigcup \Gamma_2 \bigcup (0, 0)$. Positive and negative orbits of (1.1) passing through $p \in \mathbb{R}^2$ are shown by $\mathrm{O}^+(p)$ and $\mathrm{O}^-(p)$, respectively.

The following definitions are presented to state our main results.

Definition 1.2. System (1.1) has property (Z_1^+) (resp., (Z_3^+)) if there exists a point $p(x_0, y_0) \in \Gamma_1$ (resp., $p(x_0, y_0) \in \Gamma_2$), such that the $\mathrm{O}^+(p)$ of (1.1) starting at p approaches the origin through only the first (resp., third) quadrant (see Fig. 1.2).

Definition 1.3. System (1.1) has property (Z_2^-) (resp., (Z_4^-)) if there exists a point $p(x_0, y_0) \in \Gamma_2$ (resp., $p(x_0, y_0) \in \Gamma_1$), such that the $\mathrm{O}^-(p)$ of (1.1) starting at p approaches the origin through only the second (resp., fourth) quadrant.

If system (1.1) has both properties (Z_1^+) and (Z_2^-), then a homoclinic orbit exists in the upper half-plane. Similarly, if system (1.1) has both properties (Z_3^+) and (Z_4^-), then a homoclinic orbit exists in the lower half-plane.

In the next section an implicit necessary and sufficient condition and some explicit sufficient conditions are provided for system (1.1) to have property

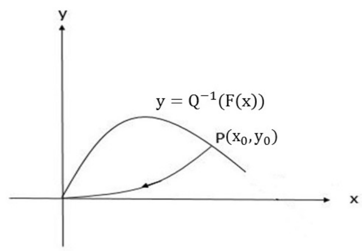

FIGURE 2. Property (Z_1^+)

(Z_1^+). Since some nonlinear functions are added to the classical Liénard system in this article, our results are proper extensions of the known ones in [1], [2], [3] and [4].

2. Necessary and Sufficient Conditions for Property of (Z_1^+)

In this section we will give necessary and sufficient conditions for system (1.1) to have properties (Z_1^+) and (Z_2^-). First, consider the following lemma about asymptotic behavior of solutions of (1.1).

Lemma 2.1. *For each point $H(c, Q^{-1}(F(c)))$ with $c > 0$ or $c < 0$, the positive or negative semi-orbit of (1.1) starting at H crosses the negative y-axis if the following condition hold.*
($\mathbf{A}_1$) *There exists a $\delta > 0$ such that $F(x) < 0$ for $-\delta < x < \delta$ or $F(x)$ has an infinite number of positive zeroes clustering at $x = 0$.*

Hereafter we assume that there exists a $\delta > 0$ such that $F(x) > 0$ for $-\delta < x < \delta$.

Theorem 2.2. *System (1.1) has property (Z_1^+) if and only if there exist a constant $\delta > 0$ and a continuous function $\phi(x)$ such that*

$$0 \le \phi(x) < F(x) \quad \text{and} \quad \int_0^x \frac{-g(\eta)}{P(\phi(\eta) - F(\eta))} d\eta \le Q^{-1}(\phi(x)) \tag{2.1}$$

for $0 < x < \delta$.

Remark 2.3. For $P(u) = u$, Theorem 2.2 gives the corresponding result of Sugie in [4].

Corollary 2.4. *Suppose that there exists $k \in (0, 1)$ and $\delta > 0$ such that*

$$\frac{1}{Q^{-1}(kF(x))} \int_0^x \frac{-g(\eta)}{P((k-1)F(\eta))} d\eta \le 1 \quad for \quad 0 < x < \delta. \tag{2.2}$$

Then, system (1.1) has property (Z_1^+).

Remark 2.5. For $P(u) = u$ and $Q(y) = y$ and taking $k = \frac{1}{2}$, Corollary 2.4 gives the result of Hara and Yoneyama in [2].

3. Homoclinc Orbit

In this section some results will be presented about the existence of homoclinic orbit in the upper half-plane for system (1.1).

Theorem 3.1. *System (1.1) has homoclinic orbit in the upper half-plane if and only if there exist a constant $\delta > 0$ and a continuous function $\phi(x)$ such that*

$$0 \leq \phi(x) < F(x) \quad \text{and} \quad \int_0^x \frac{-g(\eta)}{P(\phi(\eta) - F(\eta))} d\eta \leq Q^{-1}(\phi(x)) \tag{3.1}$$

for $0 <| x |< \delta$.

The following corollary are obtained from Theorem 3.1, which provide explicit conditions for system (1.1) to have homoclinic orbit in upper half-plane. Note that, it is assumed that there exists a $\delta > 0$ such that $F(x) > 0$ for $-\delta < x < \delta$.

Corollary 3.2. *Suppose that there exist $k \in (0, 1)$ and $\delta > 0$ such that*

$$\frac{1}{Q^{-1}(kF(x))} \int_0^x \frac{-g(\eta)}{P((k-1)F(\eta))} d\eta \leq 1 \quad for \quad 0 <| x |< \delta. \tag{3.2}$$

Then, system (1.1) has homoclinic orbit in the upper half-plane.

Remark 3.3. Suppose that F is an even and g is an odd function. It is easy to see that system (1.1) has property (Z_1^+) if and only if it has property (Z_2^-). Therefore, if system (1.1) has property (Z_1^+), then it has a homoclinic orbit in the upper half-plane.

Example 3.4. Consider the following Gause-type Predator-Prey system

$$\begin{aligned} \dot{u} &= ur(u) - vsf(u) \\ \dot{v} &= v(q(u) - D), \end{aligned} \tag{3.3}$$

with $f(u) = u$, $r(u) = \beta - \gamma|u - \alpha|$, $q(u) = u^2$, $D = \alpha^2$ and $\beta > \alpha\gamma$. System (3.3) has the positive equilibrium $E^* = (\alpha, \beta)$. By the change of variables

$$x = u - \alpha, \quad y = \ln\beta - \ln v \quad and \quad dt = uds,$$

system (3.3) will be transformed into system (1.1) with

$$P(u) = u, \quad Q(y) = \beta(1 - e^{-y}), \quad F(x) = \gamma|x|, \quad g(x) = x + \alpha - \frac{\alpha^2}{x + \alpha}.$$

Functions $F(x)$ and $g(x)$ are defined on $(-\alpha, +\infty)$ and satisfy $F(0) = 0$ and $xg(x) > 0$ for $x \neq 0$. Also, $Q(y)$ is defined on $\mathbb{R}$ satisfying $Q(0) = 0$ and $yQ(y) > 0$ for $y \neq 0$. The inverse function of $Q(y)$ is $Q^{-1}(y) = \ln(\frac{\beta}{\beta - y})$ where defined on $(-\infty, \beta)$. For $0 < x < \frac{\beta}{k\gamma}$, by using Corollary 3.2 and choosing $k = \frac{1}{2}$, it can be concluded that

$$\frac{1}{(1-k)Q^{-1}(kF(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta < \frac{8\beta}{\gamma^2}.$$

If $0 < 8\beta \leq \gamma^2$, then

$$\frac{1}{(1-k)Q^{-1}(kF(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta < 1.$$

By a similar argument, it can be shown that for $-\alpha < x < 0$

$$\frac{1}{(1-k)Q^{-1}(kF(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta < 1.$$

Therefore, by Corollary 3.2 this system has a homoclinic orbit in the upper half-plane (see Fig. 4.1).

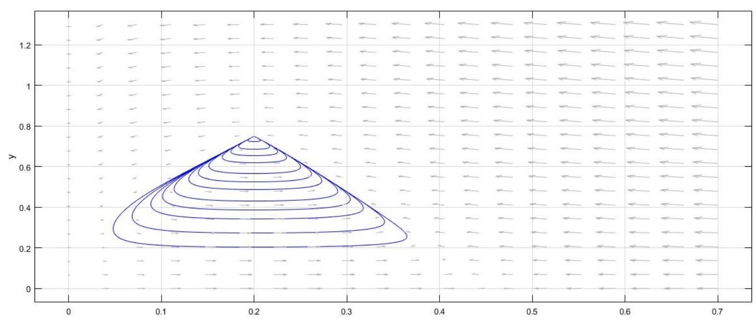

FIGURE 3. Phase portrait for system (3.3) with $\alpha = 0.2, \beta = 0.75$ and $\gamma = 3$.

Remark 3.5. Sugie and Kimoto in [5], under the assumption $Q(y) \leq my$ for $y > 0$, showed that system (1.1) with functions in (**??**) has homoclinic orbits in the upper half-plane if $0 < 8\beta \leq \gamma^2$. In this work, the existence of homoclinic orbits has been presented without the assumption $Q(y) \leq my$ for $y > 0$.

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MATHEMATICAL ANALYSIS OF HIV VIRAL INFECTION MODEL WITH LOGISTIC GROWTH RATE AND CELL-TO-CELL AND CELL-FREE TRANSMISSIONS

TOHID KASBI GHARAHASANLOU ∗AND VAHID ROOMI

Department of mathematics and statistics, Faculty and Research Institute of Basic Sciences, Imam Hossein University,
Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran,
t.kasbi@ihu.ac.ir; roomi@azaruniv.ac.ir

Abstract. It is well known that dynamical systems are very useful tools to study the viral disease such as HIV, HBV, HCV, Ebola and Influenza. This paper deals with a mathematical model of the cell-to-cell and the cell-free spread of HIV with both linear and nonlinear functional responses and logistic target cell growth. The reproduction number of each mode of transmission has been calculated and their sum has been considered as the basic reproduction number. Based on the values of the reproduction number, the local and global stability of the rest points have been investigated.

1. Introduction

Over decades, human societies have been affected by human immunodeficiency virus (HIV). HIV viruses attack the body's immune system and destroy a type of target cells known as $CD4^+$ T-cells. Studies have shown that HIV infection in humans came from a type of chimpanzee in Africa. Today, HIV infection is a contagious disease that can be transmitted from person to person. If HIV is not treated, it can lead to acquired immunodeficiency syndrome (AIDS). Unfortunately, there is currently no effective cure

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∗ Speaker.

and only with proper medical care patients may have a better quality of life. In recent years, some mathematical models have been proposed to investigate the distribution of disease and to describe epidemic illnesses related to AIDS ([1]).

Song and Neumann in [5] studied the spread of HIV by model

$$\begin{aligned}\frac{dT(t)}{dt} &= s - dT + aT(1 - \frac{T}{T_M}) - \frac{\beta TV}{1+\alpha V},\\ \frac{dI(t)}{dt} &= \frac{\beta TV}{1+\alpha V} - \delta I,\\ \frac{dV(t)}{dt} &= pI - cV,\end{aligned} \tag{1.1}$$

where $T(t)$, $I(t)$ and $V(t)$ represent the number of target cells, the number of infected cells and viral load of the virus, respectively. δ is the loss rate constant of infective cells, p is virus production rate for infected cell, c is the clearance rate constant of free viruses, s represents the rate at which new T cells are created from the source within the body, rate of infection is given by βTV, a is the maximum proliferation rate of target cells, T_M is the population density at which proliferation shuts off, d is the death rate of T cells and $\alpha > 0$ is constant for saturated mass action.

Motivated by the works of Lai and Zou in [4] and Song and Neumann in [5], in the present work, we shall study the following model of HIV infection with logistic target cell growth and two predominant transmission. Using the same notations as in [5], we investigate the model

$$\begin{aligned}\frac{dT(t)}{dt} &= s - dT + rT(1 - \frac{T}{T_M}) - \frac{b_1TV}{1+aV} - b_2TI,\\ \frac{dI(t)}{dt} &= \frac{b_1TV}{1+aV} + b_2TI - \delta I,\\ \frac{dV(t)}{dt} &= hI - lV,\end{aligned} \quad . \tag{1.2}$$

The rest of the paper is organized as follows. Section 2 deals with some basic results e.g., boundedness and non-negativity of the solutions, the basic reproduction number and the existence of equilibria. The stability of the equilibria are considered in section 3. Some of the results are illustrated numerically in section 4.

2. Equilibria and basic results

In this section, the basic properties of the solutions of (1.2) will be presented. There exists an infection-free equilibrium $E_1(T_1, 0, 0)$ where

$$T_1 = \frac{T_M}{2r}\left[r - d + \sqrt{(r-d)^2 + \frac{4rs}{T_M}}\right],$$

which represents the state of system (1.2) without viruses.

Theorem 2.1. *Starting from non-negative initial points, all solutions of (1.2) exist for all $t > 0$ and remain bounded and non-negative.*

To state our main results, the following definition will be needed.

Definition 2.2. The basic reproduction number $\mathbf{R_0}$ is defined as the expected number of secondary infections produced by an index case in a completely susceptible body cells.

According to the concept of next-generation matrix in Diekmann et al. ([2]) and the production number presented in van den Driessche and Watmough ([6]), we can compute the basic reproduction number of (1.2) as

$$\mathbf{R_0} = \mathbf{R_{01}} + \mathbf{R_{02}}, \quad \textit{where} \quad \mathbf{R_{01}} = \frac{b_1 h}{l\delta} T_1, \quad \mathbf{R_{02}} = \frac{b_2}{\delta} T_1,$$

where $T_1 = \big(\frac{r - d + \sqrt{\Delta}}{2r}\big) T_M$ and $\Delta = (r-d)^2 + \frac{4rs}{T_M}$.

By the values of $\mathbf{R_0}$, the local and the global stability of the equilibrium points of (1.2) will be studied in the next sections.

In the following, a theorem about the existence of the rest points of (1.2) will be presented.

Theorem 2.3. *System (1.2) has a unique infection-free equilibrium $E_1(T_1, 0, 0)$ if $\mathbf{R_0} \le 1$. Except for E_1, if $\mathbf{R_0} > 1$, then (1.2) has a unique positive (endemic) equilibrium $E_2(T_2, I_2, V_2)$ with $T_2 \in (0, T_1)$ where*

$$I_2 = \frac{l}{h} V_2 \quad \textit{and} \quad V_2 = \frac{1}{a} \frac{\mathbf{R_0} T_2 - T_1}{T_1 - \mathbf{R_{02}} T_2}. \tag{2.1}$$

Remark 2.4. By attention to (2.1), it can be concluded that the infected equilibrium $E_2(T_2, I_2, V_2)$ exists if and only if $\mathbf{R_0} T_2 > T_1 > \mathbf{R_{02}} T_2$.

3. Stability of Model

In this section, the local asymptotic stability of equilibria of (1.2) will be considered. Next, under certain conditions, the global asymptotic stability of E_1 will be investigated.

Theorem 3.1. *If $\mathbf{R_0} < 1$, then the infection-free equilibrium E_1 is locally asymptotically stable. If $\mathbf{R_0} > 1$, then E_1 is unstable.*

Theorem 3.2. *If $\mathbf{R_0} < 1$, then $E_1(T_1, 0, 0)$ is globally asymptotically stable.*

In the sequel, the global stability of E_2 will be presented.

Theorem 3.3. *Suppose that $\mathbf{R_0} > 1$. Then, the endemic equilibrium E_2 is globally asymptotically stable.*

Proof. Define a Lyapunov function as

$$\begin{aligned} L(T, I, V) = {} & T - T_2 - T_2 \ln \frac{T}{T_2} + I - I_2 - I_2 \ln \frac{I}{I_2} \\ & + \frac{b_1 T_2 V_2}{h I_2 (1 + a V_2)} \Big(V - V_2 - V_2 \ln \frac{V}{V_2} \Big). \end{aligned}$$

Computing the derivative of $L(T, I, V)$ along the positive solutions of (1.2), it can be written that

$$\frac{dL}{dt}\mid_{(3)}=\left(1-\frac{T_2}{T}\right)\dot{T}+\left(1-\frac{I_2}{I}\right)\dot{I}+\frac{b_1T_2V_2}{hI_2(1+aV_2)}\left(1-\frac{V_2}{V}\right)\dot{V}. \tag{3.1}$$

Therefore, from (3.1) and equilibrium conditions, it can be obtained that

$$\begin{aligned}\frac{dL}{dt}\mid_{(3)}= & -\left[d-r+r\left(\frac{T+T_2}{T_M}\right)\right]\frac{(T-T_2)^2}{T}\\ & -\frac{b_1T_2V_2}{1+aV_2}\left[\frac{a(V-V_2)^2}{(1+aV_2)(1+aV)V_2}\right]\\ & +\frac{b_1T_2V_2}{1+aV_2}\left[4-\frac{T_2}{T}-\frac{IV_2}{I_2V}-\frac{1+aV}{1+aV_2}-\frac{TV(1+aV_2)I_2}{T_2V_2(1+aV)I}\right]\\ & +b_2T_2I_2\left[2-\frac{T}{T_2}-\frac{T_2}{T}\right].\end{aligned} \tag{3.2}$$

By (3.2), it can be concluded that $\frac{dL}{dt}\leq 0$ for all $T, I, V > 0$. Hence, the endemic equilibrium E_2 is stable. On the other hand, $\frac{dL}{dt}=0$ if and only if $T=T_2$, $I=I_2$ and $V=V_2$. Let Ω be the largest invariant set in

$$\Psi=\{(T,I,V)\mid \dot{L}=0\}=\{E_2\}.$$

We have that $\Omega=\{E_2\}$. The global asymptotically stability of E_2 follows from LaSalle's invariance principle ([3]). $\square$

4. Numerical simulations

In this section, using the standard *Matlab* differential equations integrator for the Runge–Kutta method (ODE45), the numerical simulation of (1.2) will be studied. The stability of first equilibrium $E_1(1166.8560, 0, 0)$ can be seen in Fig. 1. It is obtained for the parametric values

$$\begin{aligned}&s=2,\quad r=0.2,\quad T_M=1200,\quad d=0.01,\quad b_1=0.0006,\\ &b_2=0.0004,\quad a=0.00005,\quad \delta=0.8,\quad h=0.15,\quad l=2.4.\end{aligned}$$

In this case, $\mathbf{R_0}=0.6291<1$ and the infection free equilibrium E_1 is asymptotically stable. Hereafter, we consider a set of parameters

$$\begin{aligned}&s=2,\quad T_M=1200,\quad d=0.01,\quad b_1=0.11,\\ &b_2=0.004,\quad a=0.00005,\quad \delta=0.8,\quad h=0.15,\quad l=2.4\end{aligned}$$

and different values of r. Our numerical analysis shows that for $r=0.3$, the endemic equilibrium $E_2(73.5672, 27.4768, 1.7173)$ is asymptotically stable (See Fig. 2). In this case, $\mathbf{R_0}=15.8619>1$ and eigenvalues of the characteristic equation are $\lambda_1=-0.0026+0.3871i$, $\lambda_2=-0.0026-0.3871i$ and $\lambda_3=-2.9460$.

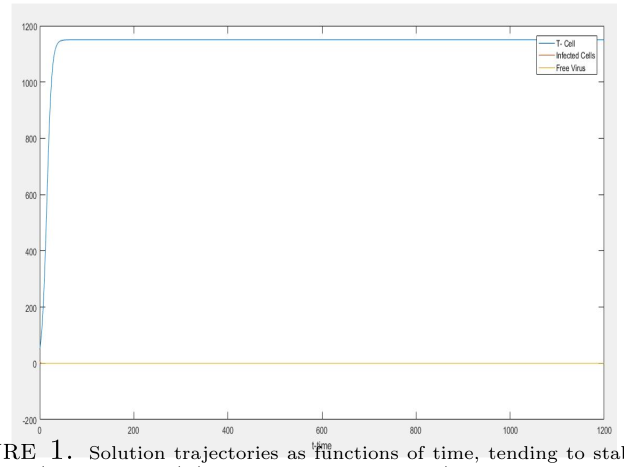

FIGURE 1. Solution trajectories as functions of time, tending to stable equilibrium $E_1(1166.8560, 0, 0)$ ($r = 0.2$, $\mathbf{R_0} = 0.6291 < 1$).

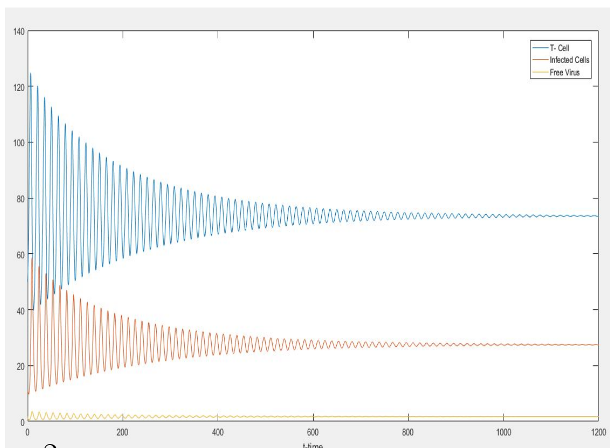

FIGURE 2. Solution trajectories as functions of time, tending to stable equilibrium $E_2(73.5672, 27.4768, 1.7173)$ ($r = 0.3$, $\mathbf{R_0} = 15.8619 > 1$).

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GENERALIZED APPROXIMATE CONVEX FUNCTIONS AND VARIATIONAL INEQUALITIES

Z. KEFAYATI∗AND M. OVEISIHA

Department of Pure Mathematics, Imam Khomeini International University, Qazvin 34194, Iran
zohrekefayati@gmail.com, oveisiha@sci.ikiu.ac.ir

Abstract. In this article, we are going to introduce sense of approximate convexity as approximate pseudoconvexity and approximate quasiconvexity for set-valued maps. Also, we consider a set-valued optimization problem and we investigate relations among solutions of this problem and solutions of generalized Minty and Stampacchia variational inequalities.

1. Introduction

The concept of convexity is very important in optimization theory as a local minimum for a convex function becomes global minimum. In order to generalize the concept of convexity Jofre et al. [4] defined the notion of convexity and by using it Ngai et al. [5] presented the concept of approximate convexity which consists of several useful and interesting properties of convex functions. Daniilidis and Georgiev [2] showed that a locally Lipschitz function is approximate convex if and only if its Clarke subdifferential is a submonotone operatore. Variational inequality theory was introduced by Hartman and Stampacchia (1966) as a tool for the study of partial differential equations with applications principally drawn from mechanics.

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Let X and Y be two Banach spaces and X^* be topological dual space of X. The norm in X and X^* will be denoted by $\| \,.\, \|$. Also, suppose that B_X is the closed unit ball of X, and $K \subset Y$ is a closed convex cone.

Definition 1.1. [3] Let $F : X \rightrightarrows Y$ be a set-valued mapping between two Banach spaces and $(\bar{x}, \bar{y}) \in grF$. Then the Normal coderivative of F at $(\bar{x}, \bar{y})$ is the set-valued mapping $D_N^* F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ $(\hat{D}^* F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*)$ given by

$$D_N^* F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* | (x^*, -y^*) \in N((\bar{x}, \bar{y}); grF)\}.$$

Definition 1.2. [1] Let $F : X \rightrightarrows Y$ be a set-valued mapping. Then the Normal subdifferential of F at the point $(\bar{x}, \bar{y}) \in epiF$ in the direction $y^* \in Y^*$ is defined by $\partial F(\bar{x}, \bar{y})(y^*) := D_N^* \mathcal{E}_F(\bar{x}, \bar{y})(y^*)$.

In the following, we present a generalization of approximate convexity for set-valued maps.

Definition 1.3. Let $\Omega \subset X$ be a convex set and $F : \Omega \subset X \rightrightarrows Y$. F is said to be approximately K-convex at $x_0 \in domF$ if for every $\alpha > 0$ there exists $\delta > 0$ (depending on x_0 and α) such that for all $x_1, x_2 \in B(x_0, \delta)$ and $t \in [0, 1]$, one has

$$tF(x_1) + (1-t)F(x_2) + \alpha t(1-t)\|x_1 - x_2\| e \subseteq F(tx_1 + (1-t)x_2) + K,$$

for a $e \in intK$ with $\|e\| = 1$.

Remark 1.4. Let $\Omega \subset X$ be a convex set and $F : \Omega \subset X \rightrightarrows Y$.

- F is said to satisfy Condition $(AC)_1$ at $x_0 \in domF$ if for every $\alpha > 0$ there exists $\delta > 0$ such that for any $x_i \in B(x_0, \delta)$, $y^* \in K^+ \cap S_{Y^*}$ and $y_i \in F(x_i)$, $(i = 1, 2)$, one has
$$< \xi, x_2 - x_1 > -\alpha \|x_2 - x_1\| \leq y^*(y_2) - y^*(y_1),$$
for some $\xi \in \partial F(x_1, y_1)(y^*)$.
- F is said to satisfy Condition $(AC)_2$ at $x_0 \in domF$ if for every $\alpha > 0$ there exists $\delta > 0$ such that for any $x_i \in B(x_0, \delta)$, $y^* \in K^+ \cap S_{Y^*}$, $y_i \in F(x_i)$, $(i = 1, 2)$, and $\xi \in \partial F(x_1, y_1)(y^*)$ one has
$$< \xi, x_2 - x_1 > -\alpha \|x_2 - x_1\| \leq y^*(y_2) - y^*(y_1).$$

2. Mean Results

In this section, we give some difinitions of generalized approximate convexity and generalized variational inequalities.

Definition 2.1. Let $\Omega \subset X$ be a convex set. A set-valued map $F : \Omega \subseteq X \rightrightarrows Y$ is said to be

- approximate pseudoconvex of type I around $x_0 \in domF$, if for all $\alpha > 0$, there exists $\delta > 0$ such that for all $x_i \in B(x_0, \delta) \cap \Omega$, $y^* \in K^+ \cap S_{Y^*}$ and $y_i \in M_{y^*}(x_i)$, $(i = 1, 2)$, if
$$< \xi, x_1 - x_2 >\geq 0, \text{ for some } \xi \in \partial F(x_2, y_2)(y^*),$$

then

$$y^*(y_1) - y^*(y_2) \geq -\alpha\|x_1 - x_2\|.$$

- approximate pseudoconvex of type II around $x_0 \in domF$, if for all $\alpha > 0$, there exists $\delta > 0$ such that for all $x_i \in B(x_0, \delta) \cap \Omega$, $y^* \in K^+ \cap S_{Y^*}$ and $y_i \in M_{y^*}(x_i)$, $(i = 1, 2)$, if

$$< \xi, x_1 - x_2 > +\alpha\|x_1 - x_2\| \geq 0, \text{ for some } \xi \in \partial F(x_2, y_2)(y^*),$$

then

$$y^*(y_1) \geq y^*(y_2).$$

Definition 2.2. Let $\Omega \subset X$ be a convex set. A set-valued map $F : \Omega \subseteq X \rightrightarrows Y$ is said to be

- approximate quasiconvex of type I around $\bar{x} \in domF$, if for all $\alpha > 0$, there exists $\delta > 0$ such that for all $x_i \in B(\bar{x}, \delta) \cap \Omega$, $y^* \in K^+ \cap S_{Y^*}$ and $y_i \in M_{y^*}(x_i)$, $(i = 1, 2)$, if

$$y^*(y_1) \leq y^*(y_2),$$

then

$$< \xi, x_1 - x_2 > -\alpha\|x_1 - x_2\| \leq 0, \quad \forall \xi \in \partial F(x_2, y_2)(y^*).$$

- approximate quasiconvex of type II around $\bar{x} \in domF$, if for all $\alpha > 0$ there exists $\delta > 0$ such that for all $x_i \in B(\bar{x}, \delta) \cap \Omega$, $y^* \in K^+ \cap S_{Y^*}$ and $y_i \in M_{y^*}(x_i)$, $(i = 1, 2)$, if

$$y^*(y_1) \leq y^*(y_2) + \alpha\|x_1 - x_2\|,$$

then

$$< \xi, x_1 - x_2 > \leq 0, \quad \forall \xi \in \partial F(x_2, y_2)(y^*).$$

Definition 2.3. A set-valued mapping $\partial F : X \times Y \times Y^* \rightrightarrows X^*$ is said to be approximate α-monotone around $\bar{x} \in domF$, if for all $\alpha > 0$, there exists $\delta > 0$ such that for each $x_i \in B(\bar{x}, \delta) \cap \Omega$, $y^* \in K^+ \cap S_{Y^*}$, $y_i \in M_{y^*}(x_i)$, $\xi \in \partial F(x_i, y_i)(y^*)$, $(i = 1, 2)$, one has

$$< \xi_2 - \xi_1, x_2 - x_1 > \geq -\alpha\|x_2 - x_1\|,$$

Now, we consider the following optimization problem:

$$\min F(x), \text{ subject to } x \in domF, \tag{2.1}$$

where $F : \Omega \subseteq X \rightrightarrows Y$.
The next two definitions are allocated solutions of problem (2.1).

Definition 2.4. A point $(\bar{x}, \bar{y}) \in grF$ is said to be a scalarized locally quasi efficient solution (SLQE) of problem (2.1) iff there exist $\alpha > 0$ and $\delta > 0$, such that for any $y^* \in K^+ \backslash \{0\}$, $x \in B(\bar{x}, \delta) \cap \Omega$ and $y \in F(x)$, one has

$$y^*(\bar{y}) \leq y^*(y) + \alpha\|x - \bar{x}\|.$$

Definition 2.5. A point $(\bar{x}, \bar{y}) \in grF$ is said to be locally weak quasi efficient solution (LWQE) of problem (2.1) iff there exist $\alpha > 0$ and $\delta > 0$, such that

$$(F(x) - \bar{y}) \cap (-\alpha\|x - \bar{x}\|e - intK) = \emptyset, \quad \forall x \in B(\bar{x}, \delta) \cap \Omega.$$

The next lemma gives a relation between (SLQE) and (LWQE).

Lemma 2.6. *Every solution of (SLQE) is a (LWQE) of problem* (2.1).

Now, we consider the following generalized Minty and Stampacchia inequality:
$(GMVIP)$: Generalized Minty variational inequality problem consists of finding a vector $\bar{x}$, such that for any $\alpha > 0$ there exists $\delta > 0$ such that for any $x \in B(\bar{x}, \delta) \cap \Omega$ and $y^* \in K^+ \cap S_{Y^*}$, there exist $y \in M_{y^*}(x)$ and $\xi \in \partial F(x, y)(y^*)$ that

$$< \xi, \bar{x} - x > \leq \alpha\|x - \bar{x}\|.$$

$(GSVIP)$: Generalized Stampacchia variational inequality problem consists of finding a vector $\bar{x}$, such that for an $\alpha > 0$, there exists $\delta > 0$, such that for each $x \in B(\bar{x}, \delta) \cap \Omega$ and $y^* \in K^+ \cap S_{Y^*}$ there exists $\bar{y} \in M_{y^*}(\bar{x})$ and $\xi \in \partial F(\bar{x}, \bar{y})(y^*)$ that

$$< \xi, x - \bar{x} > \geq -\alpha\|x - \bar{x}\|.$$

In the next theorem, we prove relation between (SLQE) and (GMVIP).

Theorem 2.7. *Suppose that* $F : \Omega \subseteq X \rightrightarrows Y$ *satisfies condition* $(AC)_2$*. If* $\bar{x}$ *is a solution of (SLQE), then it is a solution of (GMVIP).*

In the following, we investigate relation between (GSVIP) and (SLQE) by two theorems.

Theorem 2.8. *Let* $\bar{x}$ *be a solution of (GSVIP) with respect to* α *and* F *be approximate pseudoconvex of type II around* $\bar{x} \in domF$*. Then* $\bar{x} \in (SLQE)$.

Theorem 2.9. *Let* $\bar{x} \in (SLQE)$ *and* F *be approximate quasiconvex of type II around* $\bar{x}$*. Then* $\bar{x}$ *is a solution of (GSVIP) with respect to same* α.

In the finally, we prove that a solution of (GSVIP) can be a solution of (GMVIP).

Theorem 2.10. *Let* $\bar{x}$ *be a solution of (GSVIP) with respect to* α *and* ∂F *be approximate* α*-pseudomonotone. Then* $\bar{x}$ *is a solution of (GMVIP) with respect to same* α.

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Oral Presentation **$*$: Speaker**

APPROXIMATELY JENSEN-HOSSZU $\rho-$FUNCTIONAL EQUATION

VAHID KESHAVARZ* AND MOHAMMAD TAGHI HEYDARI

Department of Mathematics, College of Sciences, Yasouj University, Yasouj, 75918 Iran
v.keshavarz68@yahoo.com and heydari@yu.ac.ir

Abstract. In this paper, we introduce the concept of the Jensen-Hosszu $\rho-$functional equations between Banach algebras and we investigate it as an additive equation. Also, we prove the Hyers-Ulam stability of Jensen-Hosszu $\rho-$functional equations between Banach algebras.

1. Introduction

In 1940, Ulam[10] presented some unsolved problems, and among them posed the following question. "when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation? " Ulam raised the stability of functional equations and Hyers[2] in 1941 was the first one which gave an affirmative answer to the question of Ulam for additive mapping between Banach spaces.
In 1967, M. Hosszu introduced the functional equation $f(x + y - xy) = f(x) + f(y) - f(xy)$ in a presentation at a meeting on functional equations held in Zakopane, Poland. In honor of M. Hosszu, this equation is called Hosszu's functional equation. As one can easily see, Hosszu's functional equation is a kind of generalized form of the $f(x + y) = f(x) + f(y)$ functional equation. In 1996, L. Losonczi [9] proved the Hyers-Ulam stability of

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* Sspeaker.

the Hosszu equation in the class of real functions and expressed the following theorem.

Theorem 1.1. *(L. Losonczi) Let E be a Banach space and suppose that $f : E \to E$ satisfies the inequality*

$$\|f(x+y-xy) - f(x) - f(y) + f(xy)\| \leq \delta \quad (x, y \in \mathbb{R})$$

for some $\delta \geq 0$. Then there exists an additive function $T : E \to E$ and a unique constant $b \in E$ such that

$$\|f(x) - T(x) - b\| \leq 20\delta$$

for all $x \in E$.

Many authors have searched the stability of Cauchy, Jensen and Hosszu equation based on the concept of Hyers-Ulam stability (see [1, 3, 4, 5, 6, 7, 8]).

In the following we defined Jensen-Hosszu $\rho-$functional equation on Banach algebras.
Let $\mathfrak{A}$ and $\mathfrak{B}$ are two Banach algebras, let a mapping $f : \mathfrak{A} \to \mathfrak{B}$ satisfies

$$f(x+y-xy) + f(xy) - 2f(\frac{x+y}{2}) = \rho\Big(f(x+y-xy) + f(xy) - f(x+y)\Big) \tag{1.1}$$

and

$$f(x+y-xy) + f(xy) - f(x+y) - \rho\Big(f(x+y-xy) + f(xy) - 2f(\frac{x+y}{2})\Big) \tag{1.2}$$

where $\rho \neq 0, \pm 1$ is a fixed real number and for all $x, y \in \mathfrak{A}$, then we called Jensen-Hosszu $\rho-$functional equation.
In this work, we solve the functional equations of the form (1.1) and (1.2) in the class of real functions as an additive equation and prove them with the above ideas theorems have stable in the Hyers–Ulam sense.

2. Stability of Jensen-Hosszu $\rho-$functional equation

In this section, let $\mathfrak{A}$ and $\mathfrak{B}$ are two Banach algebras. Firstly, in the next lemma, we solve that f is an additive mapping.

Lemma 2.1. *If a mapping $f : \mathfrak{A} \to \mathfrak{B}$ satisfies*

$$f(x+y-xy) + f(xy) - 2f(\frac{x+y}{2}) = \rho\Big(f(x+y-xy) + f(xy) - f(x+y)\Big) \tag{2.1}$$

for all $x, y \in \mathfrak{A}$ and $\rho \neq 0, \pm 1$ is a fixed real number, then the mapping f is an additive equation.

In the following theorem, the functional equation (1.1) can be stable in Banach algebras.

Theorem 2.2. *Let $f : \mathfrak{A} \to \mathfrak{B}$ be a mapping such that*

$$\Big\| f(x+y-xy)+f(xy)-2f(\frac{x+y}{2})-\rho\Big(f(x+y-xy)+f(xy)-f(x+y)\Big)\Big\| \leq \delta \tag{2.2}$$

where $\rho \neq 0, \pm 1$ is a fixed real number, for some $\delta \geq 0$ and for all $x, y \in \mathfrak{A}$. Then there exists a unique additive $T : \mathfrak{A} \to \mathfrak{B}$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\delta$$

for all $x \in \mathfrak{A}$.

In the next lemma, we solve that f is an additive mapping.

Lemma 2.3. *If a mapping $f : \mathfrak{A} \to \mathfrak{B}$ satisfies*

$$f(x+y-xy)+f(xy)-f(x+y) = \rho\Big(f(x+y-xy)+f(xy)-2f(\frac{x+y}{2})\Big) \tag{2.3}$$

for all $x, y \in \mathfrak{A}$ and $\rho \neq 0, \pm 1$ is a fixed real number, then the mapping f is an additive equation.

In the following theorem, we investigate Hyers-Ulam staility of functional equation (1.2) in Banach algebras.

Theorem 2.4. *Let $f : \mathfrak{A} \to \mathfrak{B}$ be a mapping such that*

$$\Big\| f(x+y-xy)+f(xy)-f(x+y)-\rho\Big(f(x+y-xy)+f(xy)-2f(\frac{x+y}{2})\Big)\Big\| \leq \delta \tag{2.4}$$

where $\rho \neq 0, \pm 1$ is a fixed real number, for some $\delta \geq 0$ and for all $x, y \in \mathfrak{A}$. Then there exists a unique additive $T : \mathfrak{A} \to \mathfrak{B}$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in \mathfrak{A}$.

3. Conclusions

We introduced the new concept of Jensen-Hosszu ρ-functional equations in Banach algebras and in the lemmas we investigated it as an equal to an additive equation and in the main theorems, we proved that the Jensen-Hosszu ρ-functional equations can be stable in Banach algebras.

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Oral Presentation **∗ : Speaker**

ANALYSIS OF A NONLINEAR OPERATOR IN THE FRAME OF EVOLUTIONARY GAME FOR DUAL RANDOM MARKETS

MAHDI KESHTKAR∗, ELYAS SHIVANIAN

Department of Mathematics, Buein Zahra Technical University, Buein Zahra, Qazvin, Iran
keshtkarmahdi@gmail.com
Department of Applied Mathematics, Imam Khomeini International University, Qazvin 34148-96818, Iran
shivanian@sci.ikiu.ac.ir

Abstract. We study the evolutionary dynamic of population, composed of two kinds of individuals, distributed randomly. In this game, evolution takes place in different periods of time between individuals frequently. We present a continuous nonlinear operator, which describes and fulfills this discrete time evolutionary game. Furthermore, we set up the adequate mathematical framework to obtain the fixed point of this operator. Based on this equilibrium state of the evolutionary operator, it is proved the possessions of the individuals vanish.

1. Introduction

The population is composed of individuals which can have two differing "life strategies", and the success or failure of these strategies has a direct consequence upon the continued reproductive success of the individuals[1]. In the last years, some different techniques and models, from statistical

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∗ Speaker.

physics, have been applied successfully to some real data observed in economy [2]. Different models have been used to explain the origin of these wealth distributions. A kind of models considering this unknowledge associated to markets are the gas-like models [3]. In order to explain the two different types of statistical behavior before mentioned, different gas-like models have been proposed. On the one hand, the exponential distribution can be obtained by supposing a gas of agents that trade with money in binary collisions, or in first-neighbor interaction, and where the agents are selected in a random, deterministic or chaotic way [4].
Alongside this approach, however, there has also emerged a significant literature that seeks to extend such evolutionary dynamics to games with nonlinear operators [5]. This paper seeks to address further this lacuna and provide foundations to the nonlinear operators for games with two kinds of player (or equivalently agent) that compete together for obtaining wealth. We construct an operator that governs the discrete time evolution of the wealth distribution in population that is composed of two kinds of individuals, say type F (Fight) and type Y (Yield), distributed randomly which interact by pairs and exchange their money in a random way. It is shown the previous model [6] describing exponential wealth distribution in a random market is the special case of present model if we omit the individuals of type F (Fight).
The rest of the paper is structured as follows: In section 2, we introduce the nonlinear model for population games with two kind of players, section 3 establishes the fundamental mathematical properties of the operator T. Finally, a conclusion is given in section 4.

2. Nonlinear model

We consider an ensemble of economic agents (individuals or equivalently players) in two categories (Yields and Fights) which trading their money by pairs in a random manner. There are three kinds of trading between players (individuals) which are as follows by first, second and third cases. Before that, in all following cases, we notice that ε is a number in the interval $(0,1)$. Moreover, the agents $(i,\ j)$ are randomly chosen. Finally, their initial money (m_i, m_j), at time$t$, is transformed after the interaction into $\left(m_i', m_j'\right)$at time $t+1$.
The continuous version of this model considers the evolution of an initial wealth distribution $p_0(m)$ at each time step n under the action of an operator T. Thus, the system evolves from time n to time $n+1$ to asymptotically reach the equilibrium state of zero wealth, i.e.

$$\lim_{n\to\infty} T^n(p_0(m)) \to 0 \tag{2.1}$$

The derivation of the operator $T$is as follows [6]. Suppose that $p_n$is the wealth distribution in the ensemble at time n. The probability to have a quantity of money x at time $n+1$ will be the sum of the probabilities of all those pairs of agents (u, v) able to produce the quantity x after their interaction, that is, all the pairs verifying $u+v > x$ in the first case and third case and all the pairs verifying $u + v - c > x$ in the second case. Thus, the probability that two of these agents with money (u, v) interact between them is $p_n(u)*p_n(v)$. Their exchange is totally random and then they can give rise with equal probability to any value $x$comprised in the interval $(0, u + v)$ for the first and third cases, and $(0, u + v)$ for the second case. Therefore, the probability to obtain a particular x (with $x < u+v$ *or* $x+c < u+v$) for the interacting pair will be $\frac{p_n(u)*p_n(v)}{u+v}$. Then, we present the general continuous nonlinear operator for discrete time evolutionary game (it is more general than that considered in [8-10]), which comes from the combination of the above three cases, as following form

$$\begin{aligned} p_{n+1}(x) &= Tp_n(x) \\ &= \tfrac{1}{3} \iint_{u+v>x} \tfrac{p_n(u)p_n(v)}{u+v} dudv + \tfrac{1}{3} \iint_{u+v>x} \tfrac{p_n(u)p_n(v)}{u+v} dudv \\ &+ \tfrac{1}{3} \iint_{u+v>x+c} \tfrac{p_n(u)p_n(v)}{u+v} dudv \end{aligned} \tag{2.2}$$

The right hand terms are related to cases (Yield, Yield), (Fight, Yield) and (Fight, Fight), respectively. if we assume $c = 0$ where we look to Yield and Fight as the same player.

3. Mathematical properties of the operator

Definition 3.1. We introduce the space L_1^+ of positive functions (wealth distributions) in the interval $[0, \infty)$,

$$L_1^+[0, \infty) = \{y : [0, \infty) \to R^+ \cup \{0\}, \|y\| < \infty\}, \tag{3.1}$$

with norm-1

$$\|y\| = \int_0^\infty y(x)dx. \tag{3.2}$$

In particular, consider the subset of $L_1^+[0, \infty)$ i.e. the unit sphere

$$B = \left\{y \in L_1^+[0, \infty), \quad \|y\| = 1\right\}.$$

Definition 3.2. For $x \geq 0$and $y \in L_1^+[0, \infty)$ the action of operator Ton y is defined by

$$\begin{aligned} T(y(x)) &= \tfrac{1}{3} \iint_{S(x)} \tfrac{y(u)y(v)}{u+v} dudv + \tfrac{1}{3} \iint_{S(x)} \tfrac{y(u)y(v)}{u+v} dudv \\ &+ \tfrac{1}{3} \iint_{S_c(x)} \tfrac{y(u)y(v)}{u+v} dudv \end{aligned} \tag{3.3}$$

where $S(x)$and $S_c(x)$ are the regions of the plane representing the pairs of agents (u, v) which can generate a richness x after their trading, i.e.

$$S(x) = \{(u, v), u, v > 0, u + v > x\}$$

$$S_c(x) = \{(u, v), u, v > 0, u + v > x + c\}$$

If there was not cost c for any of agents which are trading money, operator Tdefined in (3.3) conserves the norm $(\| \cdot \|)$, i.e. T maintains the total number of agents (those agents which are active in game) of the system, i.e.

$\|Tp\| = \|p\| = 1$, that by extension implies the conservation of the total richness of the system. However, in the present model, T does not maintain the total numbers of agents, which are active in game, because some of them lose their total money, in the other word, their money vanish. Therefore, it is credible to expect $\|Tp\| < \|p\|$.

Lemma 3.3. *We claim that for anyy $\in L_1^+[0,\infty)$ and $c > 0$,*

$$c\int_c^\infty \int_c^\infty \frac{y(u)y(v)}{u+v}dudv < \|y\|^2 .$$

Theorem 4. *For any $y \in L_1^+[0,\infty)$ we have*

$$||T\mathrm{y}|| \leq ||y||^2 - c\int_c^\infty \int_c^\infty \frac{y(u)y(v)}{u+v}dudv.$$

It means that the number of active agents in the economic system is not conserved in time .i.e. in the unit sphere B, we observe that if $y \in B$ then $Ty \notin B$.

Theorem 3.4. *Consider the unit sphere$B = \{y \in L_1^+[0,\infty),\ \|y\| = 1\}$, if $y \in B$ then $y_{n+1}(x) = Ty_n(x) = T^n y(x)$ is a decreasing operator respected to norm-1 while it remains always $y_n(x) \in L_1^+[0,\infty)$.*

Corollary 3.5. *Suppose that $y \in B$ then the system asymptotically reach the equilibrium distribution 0, i.e.*

$$\lim_{n\to\infty} T^n(y(x)) \to 0.$$

4. Conclusion

Summarizing, in population that is composed of two kinds of individual which compete together in dual market randomly by trading their money. We have introduced a continuous nonlinear operator and then obtained the fixed point of this operator. Based on this equilibrium state of the evolutionary operator, it has been proved the possession of the individuals vanishes.

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Oral Presentation

∗ : Speaker

ON THE ANALYSIS OF A GENERAL NONLINEAR OPERATOR FOR A MODEL IN RANDOM MARKETS

MAHDI KESHTKAR*, ELYAS SHIVANIAN

Department of Mathematics, Buein Zahra Technical University, Buein Zahra, Qazvin, Iran
keshtkarmahdi@gmail.com
Department of Applied Mathematics, Imam Khomeini International University, Qazvin 34148-96818, Iran
shivanian@sci.ikiu.ac.ir

Abstract. A generalization of the continuous economic model is proposed for random markets. In this model, agents interact by pairs and exchange their money in a random way, in general, with possibly non-constant total amount of "money". This model takes the form of an iterated nonlinear map of the distribution of wealth. We show the only way to reach equilibrium fixed point distribution is the agents to share their money without expansion or contraction factor.

1. Introduction

Recently, based on the idea of pseudo-gases, a control parameter that shows the degree of exchanges between economic factors has been considered in the interval [0,1], that is, if the value of the parameter is zero, there is no interaction between the factors, and if it is one, all factors interact under these conditions. They have reached the conclusion that the Gibbs exponential distribution is established for the mentioned interval [1]. The

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relationship and behavior of the moments when they are not convergent have been investigated on the model [2].
Also, to see the convergence of exponential wealth distribution in discrete stochastic markets and their complete analysis, you can refer to reference [3]. A similar model has been proposed for ideal gases, which shows that it converges to the Maxwell distribution, based on which it has been considered as the equilibrium point of the operator with the simulations [4]. The upward growth of entropy for the model in random markets is proof [5]. Following the works, in this article we have proposed a continuous economic model for random markets. In the next section, we state a generalization of the model. Then, we discuss the features of the operator.

2. Generalization of Z-model

In this section, we suggest a general form of Z-model that is similar to the behavior of original model (Z-model) in which each of the two partners in a transaction have a random amount u and v. During the transaction, they put first the whole amount $(u+v)$ in a basket and then share its content randomly. The new model is defined as

$$P_{n+1}(x) = Tp_n(x) = \iint_{S_{a,b}(x)} dudv \frac{P_n(u)P_n(v)}{au+bv}, \tag{2.1}$$

where, a and b are real positive parameters, and $S_{a,b}(x)$ is defined by the set

$$\{(u,v),\ u,v>0,\ \ x<\ au+bv\}$$

. In this model at the time of the transaction between the two individuals, one of the individual puts au in the basket (instead of u in the Z-model) and the other puts bv in the basket, instead of v. As it will be seen, this model is not conservative except when some special conditions hold for the coefficients$a$ and b which will be determined later. If we consider the symmetrical interaction for the pair of agents(v,u), in this case the first agent will put av in the basket and the second one bu. For both trades, those of the pairs$(u,v)$and (v,u), the total money to share in the basket is $(a+b)(u+v)$.

3. Properties of the operator T

First, in order to set up the adequate mathematical framework, we provide the following definitions.

Definition 3.1. We introduce the space L_1^+ of positive functions (wealth distributions) in the interval $[0,\infty)$,

$$L_1^+[0,\infty) = \{y:[0,\infty)\to R^+\cup\{0\}, \|y\|<\infty\},$$

with norm-1

$$\|y\| = \int_0^\infty y(x)dx.$$

In particular, consider the subset of $L_1^+[0,\infty)$ i.e. the unit sphere

$$B = \left\{y \in L_1^+[0,\infty), \quad \|y\| = 1\right\}$$

Definition 3.2. We define the mean richness $< x >_y$ associated to a wealth distribution $y \in L_1^+[0,\infty)$ as the mean value of $x$for the distribution$y$. In the rest of the paper, we will represent it by$< y >$. Then,

$$< y > \equiv < x >_y = \|xy(x)\| = \int_0^\infty xy(x)dx.$$

Definition 3.3. For $x \geq 0$and$y \in L_1^+[0,\infty)$ the action of the operator Ton y is defined by

$$T(y(x)) = \iint_{S_{a,b}(x)} \frac{y(u)y(v)}{au+bv} dudv$$

where$S_{a,b}(x)$ is the region of the plane representing the pairs of agents (u,v) which can generate a richness x after their trading, i.e.

$$S_{a,b}(x) = \{(u,v), u, v > 0, \, au + bv > x\}$$

Theorem 3.4. *For any $y \in L_1^+[0,\infty)$ we have $\|Ty\| = \|y\|^2$. In particular, for y being a PDF, i.e. if $\|y\| = 1$, then $\|Ty\| = 1$. (It means that the number of agents in the economic system is conserved in time).*

Theorem 3.5. *The operator T is Lipchitz continuous in B with Lipchitz constant ≤ 2.*

Theorem 3.6. *The mean value$< y >$of a PDF y is not conserved in general, that is it would be possible$< Ty > \neq < y >$ for any $y \in B$. (It means that the mean wealth, and by extension the total richness, of the economic system are not preserved in time).*

Corollary 3.7. *The mean wealth, and by extension the total richness, of the economic system is preserved in time provided that $a + b = 2$.*

In the next section, it will be revealed that the total richness increases when $a + b > 2$ and decreases when $a + b < 2$.

Theorem 3.8. *For any$y \in L_+^1[0,\infty]$, $n\epsilon\mathbb{N}$ and $a, b \in \mathbb{R}^+$it holds*

$$< T^n y > - < y > = ((\frac{a+b}{2})^n - 1) < y > .$$

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Oral Presentation **$*$: Speaker**

SOLUTIONS TO A (P, Q)-BIHARMONIC EQUATION WITH NAVIER BOUNDARY CONDITIONS

A. KHALEGHI*, A.RAZANI

Department of Pure Mathematics, Imam Khomeini International University, Qazvin 34194, Iran
A.Khaleghi@edu.ikiu.ac.ir; razani@sci.ikiu.ac.ir

Abstract. We study the existence of weak solutions to a (p, q)-biharmonic elliptic equation involving a singular term under Navier boundary conditions, by using variational methods.

1. Introduction

Stationary problems involving singular nonlinearities, as well as the associated evolution equations, describe naturally several physical phenomena and applied economical models. This kind of problems intensively studied in the last decades, specially with the Steklov boundary conditions [4]. In the present paper, we consider the following (p, q)-biharmonic problem

$$\begin{cases} \Delta_p^2 u + \Delta_q^2 u + \theta(x)\frac{|u|^{s-2}u}{|x|^{2s}} = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N > 2)$ is a bounded domain with boundary of class C^1 and p, q are positive parameters satisfying the following inequalities $max\{2, N/2\} < q < p < +\infty$. And, $\Delta_r^2 u := \Delta(|\Delta u|^{r-2}\Delta u)$ denotes r-biharmonic operator for $r \in \{p, q\}$; $\theta \in L^\infty(\Omega)$ is a real function with

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$\inf_{x\in\overline{\Omega}}\theta(x) > 0$; s is a constant such that $1 < s < N/2$; $\lambda > 0$ is a real parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which holds the following growth condition:

$$|f(x,s)| \leqslant a_1 + a_2|s|^{\gamma-1} \tag{1.2}$$

for $(x,s) \in \Omega \times \mathbb{R}$, where a_1, a_2 and γ are positive constants such that $\gamma \leq p$ a.e. in Ω.

2. Basic definitions and preliminary results

Proposition 2.1. [3] *Let $q \leq p$, a.e. on Ω, then $L^p(\Omega) \hookrightarrow L^q(\Omega)$; moreover, there is a constant k_q such that $|u|_q \leqslant k_q|u|_p$.*

We denote the Sobolev space $W^{k,p}(\Omega)$ for $k = 1, 2$, by

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| \leq k\},$$

that in which $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1 \ldots \partial^{\alpha_N}x_N}$ where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is a multi-index with $|\alpha| = \Sigma_{i=1}^N \alpha_i$. The space $W^{k,p}(\Omega)$ with the norm

$$\|u\|_{k,p} = \Sigma_{|\alpha|\leq k}|D^\alpha u|_p$$

is a Banach separable and reflexive space. We assume that $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ which has the norm $\|u\|_{1,p} = |Du|_p$. In what follows, we set

$$X := W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega),$$

endowed with the norm $\|u\| := \int_\Omega |\Delta u|^p dx$.

Remark 2.2. The embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact; moreover, there exist constant $L > 0$ such that $|u|_\infty \leq L\|u\|$, where $|u|_\infty = \sup_{x\in\Omega} u(x)$.

The next is the classical Hardy-Rellich inequality mentioned in [2].

Lemma 2.3. *Let $1 < s < \frac{N}{2}$. Then for $u \in W_0^{1,s}(\Omega) \cap W^{2,s}(\Omega)$, one has*

$$\int_\Omega \frac{|u(x)|^s}{|x|^{2s}}dx \leq \frac{1}{\mathcal{H}}\int_\Omega |\Delta u(x)|^s dx,$$

where $\mathcal{H} := (\frac{N(s-1)(N-2p)}{s^2})^s$.

Definition 2.4. We say that function $u \in X$ is a weak solution of Problem (1.1) if $u = \Delta u = 0$ on $\partial\Omega$ and

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta v dx + \int_\Omega |\Delta u|^{q-2}\Delta u \Delta v dx + \int_\Omega \theta(x)\frac{|u|^{s-2}}{|x|^{2s}}uv dx - \lambda\int_\Omega f(x,u)v dx = 0$$

for every $v \in X$.

In the sequel, we put

$$\delta(x) = sup\,\{\delta > 0 : B(x, \delta) \subseteq \Omega\} \quad \text{and} \quad R := sup_{x\in\Omega}\delta(x).$$

Obviously, there exists $x^0 = (x_1^0, \cdots, x_N^0) \in \Omega$ such that $B(x^0, R) \subseteq \Omega$.

3. Existence result

Let $\Phi : X \to \mathbb{R}$ be a functional defined by

$$\Phi(u) = \frac{1}{p}\int_\Omega |\Delta u|^p dx + \frac{1}{q}\int_\Omega |\Delta u|^q dx + \frac{1}{s}\int_\Omega \theta(x)\frac{|u(x)|^s}{|x|^{2s}}dx,$$

Remark 3.1. Under the above assumptions, we gain

$$\frac{1}{p}\|u\|^p \leq \Phi(u) \leq K(\|u\|^p + \|u\|^s)$$

where $K = \max\{\frac{2}{s}, \frac{2|\theta|_\infty}{\mathcal{H}s}\}$.

Φ is continuously Gâteaux differentiable functional; moreover,

$$\langle\Phi'(u), v\rangle = \int_\Omega (|\Delta u|^{p(x)-2}\Delta u\Delta v + |\Delta u|^{q(x)-2}\Delta u\Delta v + \theta(x)\frac{|u(x)|^{s-2}uv}{|x|^{2s}})dx$$

for $u, v \in X$ (see [5]). Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with the growth condition (1.2) and define $F(x, t) := \int_0^t f(x, s)ds$. Then the functional $\Psi : X \to \mathbb{R}$ with $\Psi(u) := \int_\Omega F(x, u(x))dx$ for every $u \in X$ is continuously Gâteaux differentiable with the following compact derivative $\langle\Psi'(u), v\rangle := \int_\Omega f(x, u(x))v(x)dx$, for every u, v in X (see [5]). Now, define $I_\lambda = \Phi - \lambda\Psi$.

Theorem 3.2. *Let $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be Carathéodory function satisfy (1.2). Assume that there exist $r > 0$ and $\delta > 0$ such that*

$$K\Big((\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^p + (\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^s\Big)m\big(R^N - (\frac{R}{2})^N\big) < r,$$

where $m := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$ is the measure of unit ball of $\mathbb{R}^N$ and Γ is the Gamma function. Then for each $\lambda \in]A, B[$, where

$$A := \frac{K\Big((\frac{2\delta N}{R^2-(\frac{R}{2})^2})^p + (\frac{2\delta N}{R^2-(\frac{R}{2})^2})^s\Big)m\big(R^N - (\frac{R}{2})^N\big)}{|\Omega|\big(a_1 L(pr)^{\frac{1}{p}} + \frac{a_2}{\gamma}L_\gamma^\gamma(pr)^{\frac{\gamma}{p}}\big)},$$

and

$$B := \frac{r}{|\Omega|\big(a_1 L(pr)^{\frac{1}{p}} + \frac{a_2}{\gamma}L^\gamma(pr)^{\frac{\gamma}{p}}\big)},$$

Problem (1.1) admits at least one non-trivial weak solution.

Proof. For the given $\lambda > 0$, the functional I_λ satisfies the $(P.S.)^{[r]}$ condition. Let the function $w_\lambda \in X$ be defined by

$$w_\lambda(x) := \begin{cases} 0 & x \in \Omega \setminus B(x^0, R), \\ \delta & x \in B(x^0, \frac{R}{2}), \\ \frac{\delta}{R^2 - (\frac{R}{2})^2}(R^2 - \sum_{i=1}^N (x_i - x_i^0)^2) & x \in B(x^0, R) \setminus B(x^0, \frac{R}{2}), \end{cases} \tag{3.1}$$

where $x = (x_1, \cdots, x_N) \in \Omega$. Then,

$$\sum_{i=1}^N \frac{\partial^2 w}{\partial x_i^2}(x) = \begin{cases} 0 & x \in \big(\Omega \setminus B(x^0, R)\big) \cup B(x^0, \frac{R}{2}), \\ -\dfrac{2\delta N}{R^2 - (\frac{R}{2})^2} & x \in B(x_0, R) \setminus B(x^0, \frac{R}{2}). \end{cases}$$

So, by applying Remark 3.1, one has

$$\begin{aligned} \frac{1}{p^+}(\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^p m(R^N - (\frac{R}{2})^N) & \\ &< \Phi(w) \\ &\leq K\Big((\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^p + (\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^s\Big) m\big(R^N - (\frac{R}{2})^N\big), \end{aligned}$$

then, we gain $\Phi(w) < r$. Using Remark 3.1, for each $u \in \Phi^{-1}((-\infty, 1[)$, we have

$$\|u\| \leq [p^+\Phi(u)]^{\frac{1}{p}} \leq (p^+ r)^{\frac{1}{p}}. \tag{3.2}$$

Hence, from (3.2) and (1.2), we deduce

$$\sup_{\Phi(u) < r} \Psi(u) \leq |\Omega| \big(a_1 L(pr)^{\frac{1}{p}} + \frac{a_2}{\gamma} L^\gamma (pr)^{\frac{\gamma}{p}}\big).$$

Then, from boundedness Φ, one has

$$\frac{\Psi(w)}{\Phi(w)} > \frac{|\Omega|\big(a_1 L(pr)^{\frac{1}{p}} + \frac{a_2}{\gamma} L^\gamma (pr)^{\frac{\gamma}{p}}\big)}{K\Big((\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^p + (\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^s\Big) m\big(R^N - (\frac{R}{2})^N\big)}$$

So, by critical points results duo to Bonanno (Theorem 3.4 of [1]), for each $\lambda \in]A, B[$ the functional I_λ has at least one non-zero critical point which is the weak solution of Problem (1.1). □

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Oral Presentation

APPLICATIONS OF TENSOR ANALYSIS TO COMPUTE THE CURVATURE AND TORSION FOR IMPLICIT CURVES

MEHDI KHALEGHI*

Department of Mechanical Engineering , Sharif University of Technology ,Tehran, Iran
khaleghi63@gmail.com

Abstract. In the this paper, curvature and torsion formulas will be computed for an implicit curve in $(n+1)$-dimension by using tensor analysis and operations. Then, Goldman's results for computing the torsion of an implicit curve have been extended in $\mathbb{R}^{n+1}$ Euclidean space. In addition, some useful formulas to calculate the higher order analogues of the torsion in $(n+1)$-dimensions will be derived in this paper, using tensor operations.

1. Introduction

Curvature formulas of surfaces and curves in Euclidean space have been developed by many mathematicians so far by using differential geometry. The differential geometry of curves and surfaces can be found in textbook such as in Spivak(1975) and Stocker(1969).T. Maekawa and N.H. Patrikalakis (2001) presented Ferenet-Serret formulaes for space curves. Also they spoke about principal curvatures of explicit surface. Bajaj and Kim (1991) and B. Linn (1997) presented a formula to compute the curvature for an implicit plannar curve. R. Osserman considered some relations between sectional curvatures and the scalar curvature in (n)-dimensional Euclidean space. Klingenberg (1978) provided a curvature formula for curves which are

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* Sspeaker.

intersections of some equations in $\mathbb{R}^3$, $\mathbb{R}^4$ and $\mathbb{R}^n$. K. Nomizu worked on certain conditions to drive the tensor of curvature for hypersurfaces. Curvature formulas to calculate mean and Gaussian curvatures for arbitrary surfaces provided by Turkiyyah(1997) and Belyaev(1998). P. Hartman and L. Nirenberg considered some no change properties of hypersurfaces of dimension n immersed in $(n+1)$-dimensional Euclidean space. Different formulas to calculate the curvature of intersection curves in (3)-dimensional Euclidean space by using implicit function theorem were given by Hartmann(1996).

H. Schlichtkrull (2011) provided some formulas to calculate geodesic and normal curvatures for an arbitrary curve and relation between components of Reimann curvature tensor and the second fundamental form of implicit and explicit surfaces. Osherand Fedkiv(2003) computed some formulas to calculate the curvature for implicit curves and surfaces by using Level set method. R. Goldman (2005) found formulas to compute the curvature of curves in $(n+1)$-dimensions which were intersections of (n) hypersurfaces but for the torsion of curves, only a formula in $\mathbb{R}^3$ was driven. Formulas to calculate first, second and third curvatures of intersection curves in $\mathbb{R}^4$ were provided with O. Alessio (2009) by using implicit function theorem. X. Ye and T. Maekawa. Mohamed. S. Lone and O.Alessio and M. H. Shahid (2016) used some formulas to compute $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ and geodesic curvature in $\mathbb{R}^5$ but formula for higher order analogues of κ_4 was not provided. The study of curvature, torsion and higher-order analogous for implicit curves (for example see [1, 2]).

2. Main Results

It is well known from elementary geometry that a curve in $\mathbb{R}^3$ can be described by $x = x(t)$, $y = y(t)$ and $z = z(t)$. $(t_1 < t < t_2)$

The purpose of this work is to provide the curvature formula for an implicit curve in $(n+1)$-dimensions which is generated by the intersection of n implicit simultaneous equations[4].

A parameterized continuous curve in $\mathbb{R}^3$ is a continuous map $\gamma : I \to \mathbb{R}^3$, where $I \subseteq R$ is an open interval (of end points $0 < a < b < \infty$). The parametric curve is assumed to be of class 3. The implicit representation for a space curve can be expressed as intersection curve between two implicit surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$.

If the two implicit equations $F = 0$ and $G = 0$ can be solved for two of the variables in terms of the third, for example $\dot{y}$ and $\dot{z}$ in terms of $\dot{x}$, we obtain the curvature formula.This is always possible at least locally when $\dot{x}$ is not equal to zero.

Let us consider two implicit simultaneous equations which intersect each other in an arbitrary curve which lies in 3-dimensional Euclidean space. We can take first and second differential from two implicit functions to drive $\dot{y}$

, $\dot{z}$, $\ddot{y}$ and $\ddot{z}$:

$$\begin{cases} F(x,y,z) = z - f(x) = 0 \\ G(x,y,z) = z - g(y) = 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = -\dot{x}f_x + \dot{z} = 0 \\ \\ \frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt} + \frac{\partial G}{\partial z}\frac{dz}{dt} = -\dot{y}g_y + \dot{z} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \dot{y} = \dot{x}\frac{f_x}{g_y} \\ \dot{z} = \dot{x}f_x \end{cases} \Rightarrow \begin{cases} \ddot{y} = \ddot{x}\frac{f_x}{g_y} + \dot{x}^2(\frac{g_y^2 f_{xx} - f_x^2 g_{yy}}{g_y^3}) \\ \ddot{z} = \ddot{x}f_x + \dot{x}^2 f_{xx} \end{cases}$$

The curvature formula for the parametric curve γ is

$$\kappa = \frac{\|(\dot{y}\ddot{z} - \dot{z}\ddot{y})\widehat{e_x} + (\dot{z}\ddot{x} - \dot{x}\ddot{z})\widehat{e_y} + (\dot{x}\ddot{y} - \dot{y}\ddot{x})\widehat{e_z}\|}{\|\dot{x}\widehat{e_x} + \dot{y}\widehat{e_y} + \dot{z}\widehat{e_z}\|^3}$$

Gradients for implicit functions F and G are given by

$$\begin{cases} \overrightarrow{\nabla}F = \overrightarrow{P_1} = F_x\widehat{e_x} + F_y\widehat{e_y} + F_z\widehat{e_z} = -f_x\widehat{e_x} + \widehat{e_z} \\ \overrightarrow{\nabla}G = \overrightarrow{P_2} = G_x\widehat{e_x} + G_y\widehat{e_y} + G_z\widehat{e_z} = -g_y\widehat{e_y} + \widehat{e_z} \end{cases}$$

Now we compute the cross product of vectors $\overrightarrow{P_1}$ and $\overrightarrow{P_2}$:

$$\overrightarrow{u} = \overrightarrow{P_1} \times \overrightarrow{P_2} = g_y\widehat{e_x} + f_x\widehat{e_y} + f_x g_y\widehat{e_z}$$

So we can proof this formula for the curvature:

$$\kappa = \frac{\left\|\overrightarrow{u}.(\overrightarrow{\nabla}\mathbf{B}).\overrightarrow{u}\right\|}{\left\|\overrightarrow{u}\right\|^3} = \frac{\left\|(\overrightarrow{\nabla}F \times \overrightarrow{\nabla}G).(\overrightarrow{\nabla}\mathbf{B}).(\overrightarrow{\nabla}F \times \overrightarrow{\nabla}G)\right\|}{\left\|\overrightarrow{\nabla}F \times \overrightarrow{\nabla}G\right\|^3} \tag{2.1}$$

Now we introduce two new characters $\lambda_1 = \overrightarrow{u}.\mathbf{T}_1.\overrightarrow{u}$ and $\lambda_2 = \overrightarrow{u}.\mathbf{T}_2.\overrightarrow{u}$:
$\overrightarrow{u}.(\mathbf{T}_2 \bigotimes \overrightarrow{P_1} - \mathbf{T}_1 \bigotimes \overrightarrow{P_2}).\overrightarrow{u} = \lambda_2\overrightarrow{P_1} - \lambda_1\overrightarrow{P_2}$

And hence

$$\kappa = \frac{\left\|\lambda_2\overrightarrow{P_1} - \lambda_1\overrightarrow{P_2}\right\|}{\left\|\overrightarrow{u}\right\|^3} \tag{2.2}$$

After using above formulas we have

$$\left\|\overrightarrow{u}\right\| = \sqrt{\Omega_{i'j'k'l'}P_1^{i'}P_2^{j'}P_1^{k'}P_2^{l'}}, \tag{2.3}$$

$$\Omega_{i'j'k'l'} = \begin{vmatrix} \delta_{i'k'} & \delta_{i'l'} \\ \delta_{j'k'} & \delta_{j'l'} \end{vmatrix}$$

$$\Omega_{ijkl} = \begin{vmatrix} \delta_{ik} & \delta_{il} \\ \delta_{jk} & \delta_{jl} \end{vmatrix}$$

And the tensor form of the curvature may be written as

$$\kappa = \frac{\sqrt{\Omega_{ijkl}\lambda_i\lambda_k(\overrightarrow{P_j}.\overrightarrow{P_l})}}{\left\{\Omega_{i'j'k'l'}P_1^{i'}P_2^{j'}P_1^{k'}P_2^{l'}\right\}^{\frac{3}{2}}} \tag{2.4}$$

$$\lambda_b = \overrightarrow{u}.(\overrightarrow{\nabla}\overrightarrow{P_b}).\overrightarrow{u} = \varepsilon_{\alpha\beta\tau}\varepsilon_{\eta\sigma\omega}P_1^{\alpha}P_2^{\beta}P_1^{\eta}P_2^{\sigma}P_{b,\tau}^{\omega}$$

For all $i,j,k,l = 1,2, \quad i',j',k',l',\alpha,\beta,\tau,\eta,\sigma,\omega = 1,2,3$ and $b = i,k \in \{1,2\}$.

It is interesting to extend (2.2) for implicit space curves to a formula for implicit curves in $(n+1)$-dimensions that is, to curves is generated by the intersection of n hyper surfaces which lying in $\mathbb{R}^{n+1}(n = 2,3,\cdots)$.

3. Conclusions

In this work, Curvature and torsion formulas for parametric planar and space curves are derived in differential geometry. Due to the application of curve geometry in the analysis of space-time, geometric quantities in higher dimensions have been studied. Since driving closed formulas for the curvature and the torsion and also higher-order analogues of the torsion for implicit surfaces defined by the intersection of implicit equations $F_1(x_1,\cdots,x_{n+1}) = 0 \cap \cdots \cap F_n(x_1,\cdots,x_{n+1}) = 0$ leads to complicated formulas, studying the geometric quantities for implicit curves and surfaces is the main focus of many researchers. Closed formulas for the curvature in $(n+1)$-dimensions and for the torsion in 3-dimensions for implicit curves have been derived by Ron. Goldman in [3]. Finally, by using the MATLAB program to calculate the geometric values of well-known implicit curves, the obtained formulas are verified.

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Oral Presentation **$*$: Speaker**

ON CONVERGENCE OF ALTERNATING RESOLVENTS FOR A FINITE FAMILY OF PSEUDO-CONVEX FUNCTIONS

HADI KHATIBZADEH AND MARYAM MOOSAVI*

Department of Mathematics, Zanjan University, Iran
hkhatibzadeh@znu.ac.ir; maryam_moosavi@znu.ac.ir

Abstract. In this paper, the strong convergence of the Halpern type and weak convergence of the sequence generated by the product of resolvents of pseudo-convex functions are established in the setting of Hadamard spaces.

1. Introduction

Let (X, d) be a metric space. A geodesic from x to y is a map γ from the closed interval $[0, d(x, y)] \subset \mathbb{R}$ to X such that $\gamma(0) = x$, $\gamma(d(x, y)) = y$ and $d(\gamma(t), \gamma(t^{'})) = |t - t^{'}|$ for all $t, t^{'} \in [0, d(x, y)]$. The image of a geodesic path is called a geodesic segment, which is denoted by $[x, y]$ whenever it is unique. A metric space (X, d) is called a geodesic space if every two points of X are joined by a geodesic path, and X is said to be uniquely geodesic if every two points of X are joined by exactly one geodesic path. A subset C of X is said to be convex if C includes every geodesic segments joining two of its points. Let $x, y \in X$and $t \in [0, 1]$, and we write $tx \oplus (1t)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(x, z) = (1 - t)d(x, y) \quad and \quad d(z, y) = td(x, y).$$

.
A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of

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* Sspeaker.

three vertices (points in X) with unparameterized geodesic segment between each pair of vertices. For any geodesic triangle, there is comparison (Alexandrov) triangle $\bar{\triangle} \subset \mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j})$ for $i, j \in \{1, 2, 3\}$. Let $\triangle$ be a geodesic triangle in X and $\bar{\triangle}$ be a comparison triangle for $\bar{\triangle}$, then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all points $x, y \in \triangle$ and $\bar{x}, \bar{y} \in \bar{\triangle}$:

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

Let x, y and z be points in X and y_0 be the midpoint of the segment $[y, z]$; then, the $CAT(0)$ inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d(y, z). \tag{1.1}$$

Inequality (1.1) is known as the CN inequality of Bruhat and Titis [2].
A geodesic space X is said to be a $CAT(0)$ space if all geodesic triangles satisfy the $CAT(0)$ inequality. Equivalently, X is called a $CAT(0)$ space if and only if it satisfies the CN inequality. $CAT(0)$ spaces are examples of uniquely geodesic spaces, and complete $CAT(0)$ spaces are called Hadamard spaces.
In a unique geodesic metric space X, a set $A \subset X$, is called convex iff for each $x, y \in A$, $[x, y] \subset A$. A function $f : X \to]-\infty, +\infty]$ is called

(1) proper iff

The domain of f defined by $D(f) := \{x \in X : f(x) < \infty\}$ is nonempty.

(2) lower semicontinuous (for short, lsc) iff

$$\{x \in D(f) : f(x) \leq r\},$$

is closed for each $r \in \mathbb{R}$.

(3) convex iff for all $x, y \in X$ and for all $\lambda \in [0, 1]$

$$f((1 - \lambda)x \oplus \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y).$$

(4) α-weakly convex for some $\alpha > 0$ iff for all $x, y \in X$ and for all $\lambda \in [0, 1]$

$$f((1 - \lambda)x \oplus \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) + \alpha\lambda(1 - \lambda)d^2(x, y).$$

(5) quasi-convex iff for all $x, y \in X$ and for all $\lambda \in [0, 1]$

$$f((1 - \lambda)x \oplus \lambda y) \leq (1 - \lambda)f(x) \leq \max\{f(x), f(y)\}.$$

(6) pseudo-convex iff $f(y) > f(x)$ implies that there exists $\beta(x, y) > 0$ and $0 < \delta(x, y) \leq 1$ such that

$$f(y) - f((1 - \lambda)x \oplus \lambda y) \geq \lambda\beta(x, y), \quad \forall\lambda \in (0, \delta(x, y)).$$

Definition 1.1. Let $\{x_n\}$ be a bounded sequence in a geodesic metric space X. then, the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \bar{v} \in X : \limsup_{n\to\infty} d(\bar{v}, x_n) = \inf_{x\in v} \limsup_{n\to\infty} d(v, x_n).$$

Definition 1.2. A sequence $\{x_n\}$ in a Hadamard space X is said to be weakly converges to a point $\bar{v} \in X$ if $A(\{x_n\}) = \bar{v}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $w - \lim_{n\to\infty} x_n = \bar{v}$(see [1]).

The concept of weak convergence in metric spaces was first introduced and studied by Lim [6]. Kirk and Panyanak [5] later introduced and studied this concept in $CAT(0)$ spaces and proved that it is very similar to the weak convergence in Banach space setting.

2. MAIN RESULTS

Definition 2.1. Let $\mathcal{H}$ be a Hadamard space and $f_i : \mathcal{H} \to]-\infty, +\infty]$ be convex and lsc. For $\lambda > 0$, the resolvent of f of order λ at $x \in \mathcal{H}$ is defined as follows.

$$J_\lambda^f x := \mathrm{Argmin}_{y\in\mathcal{H}}\{f(y) + \frac{1}{2\lambda} d^2(x, y)\}.$$

Well-definedness of J_λ^f was proved by Jost [3] and Mayer [7]. In [4, Theorem 3.1] the authors proved that for an α-weakly convex function f, the resolvent $J_\lambda^f x$ exists for all $x \in \mathcal{H}$ and $\lambda < \frac{1}{2\alpha}$.

Theorem 2.2. *Suppose that $\mathcal{H}$ is a locally compact Hadamard space and $f_i : \mathcal{H} \to]-\infty, +\infty]$ for $i = 1, \cdots, N$ are proper, pseudo-convex functions and lsc. If $\liminf \lambda_k > 0$ and $\cap_{i=1}^N \mathrm{Argmin}(f_i) \neq \emptyset$. Then the sequence generated by*

$$x_{k+1} = J_{\lambda_k}^{f_N} \cdots J_{\lambda_k}^{f_1} x_k \tag{2.1}$$

converges to an element of $\cap_{i=1}^N \mathrm{Argmin}(f_i)$

Theorem 2.3. *Suppose that $\mathcal{H}$ is a Hadamard space and $f_i : \mathcal{H} \to]-\infty, +\infty]$ for $i = 1, \cdots, N$ are proper, convex and lsc. If $\liminf \lambda_k > 0$ and $\cap_{i=1}^N \mathrm{Argmin}(f_i) \neq \emptyset$ then the sequence (2.1) converges weakly to an element of $\cap_{i=1}^N \mathrm{Argmin}(f_i)$*

Halpern regularization of (2.1) gets a strong convergence theorem.

Theorem 2.4. *Suppose that $\mathcal{H}$ is a Hadamard space and $f_i : \mathcal{H} \to]-\infty, +\infty]$ for $i = 1, \cdots, N$ are α-weakly convex, quasi-convex and lsc. Let $\liminf \lambda_k > 0$ and $\cap_{i=1}^N \mathrm{Argmin}(f_i) \neq \emptyset$, $u \in \mathcal{H}$ is arbitrary and α_k satisfies the conditions*

- $0 \leq \alpha_k \leq 1$
- $\alpha_k \to 0 \quad as \quad k \to \infty$
- $\sum_{k=1}^{\infty} \alpha_k = +\infty$

If f_i for each $(1 \leq i \leq N)$ are pseudo-convex functions then the sequence generated by

$$x_{k+1} = \alpha_k u \oplus (1-\alpha_k) J_{\lambda_k}^{f_N} \cdots J_{\lambda_k}^{f_1} x_k$$

converges strongly to an element $\cap_{i=1}^{N} \mathrm{Argmin}(f_i)$

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Oral Presentation

NORM INEQUALITIES INVOLVING VARIOUS TYPES OF CONVEX FUNCTIONS

MOHSEN KIAN

Department of Mathematics, University of Bojnord, Bojnord 94531, Iran
kian@ub.ac.ir

Abstract. We present a general norm inequality for matrix functions, when various types of convexity are considered.

1. Introduction

Assume that $\mathcal{M}_n(\mathbb{C})$ is the C^*-algebra of all $n \times n$ complex matrices and $\mathcal{H}_n(\mathbb{C})$ is the real subspace of Hermitian matrices. A matrix $A \in \mathcal{H}_n(\mathbb{C})$ is said to be positive (semi-definite) and is denoted by $A \geq 0$ if all of its eigenvalues are non-negative. This induces a well-known partial order on $\mathcal{H}_n(\mathbb{C})$, the Löwner order:

$$A \leq B \iff B - A \geq 0 \qquad (A, B \in \mathcal{H}_n(\mathbb{C})).$$

A norm $|||\cdot|||$ on $\mathcal{M}_n(\mathbb{C})$ is called unitarily invariant if $|||UAV||| = |||A|||$ holds for every $A \in \mathcal{M}_n(\mathbb{C})$ and all unitary matrices U, V. The most famous unitarily invariant norms on $\mathcal{M}_n(\mathbb{C})$ are the classes Schatten p-norms and Ky Fan k-norms, see [2].

For a real interval J, we denote by $\mathcal{H}_n(J)$ the set of all Hermitian matrices, whose eigenvalues are contained in J. Let f be a real function defined on an interval J. For any Hermitian matrix $A \in \mathcal{H}_n(J)$, the Hermitian matrix $f(A)$ is defined via the spectral decomposition of $A = U^*\mathrm{diag}(\lambda_1, \ldots, \lambda_n)U$ by $f(A) = U^*\mathrm{diag}(f(\lambda_1), \ldots, f(\lambda_n))U$, in which $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A.

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Matrix means are known extensions of scalar means to the non-commutative setting. Let $A, B \geq 0$ and $t \in [0,1]$. The most famous matrix means are the arithmetic mean $A\nabla_t B = (1-t)A + tB$, the geometric mean $A\sharp_t B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}$ and the Harmonic mean $A!_t B = \left((1-t)A^{-1} + tB^{-1}\right)^{-1}$.

Recall that a real function $f : (0,\infty) \to (0,\infty)$ is called convex if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ holds for all $x, y \geq 0$ and every $t \in [0,1]$. It is known that [1] if f is a non-negative convex function on J, then

$$|||f(A\nabla_t B)||| \leq |||f(A)\nabla_t f(B)||| \tag{1.1}$$

holds for every unitarily invariant norm $|||\cdot|||$ and all $A, B \in \mathcal{H}_n(J)$.

We aim to generalize (1.1) for more type of convexity.

2. Main Result

The convex functions are defined by comparison of arithmetic means of points in the domain and in the image of a function. When the arithmetic function is replaced by other various means, some other types of convex functions are derived. Let α and β be two means. We say that a function f is α-β-convex when $f(\alpha(x,y)) \leq \beta(f(x), f(y))$ holds for all x, y in the domain of f. We refer the reader to [3, 5] for more information about various convexities and examples.

With respect to this notion, we present the next result, which provides a generalization of (1.1).

Theorem 2.1. *Let f be a positive function defined on $(0,\infty)$. If f is a α-β-convex function, then*

$$|||f(\alpha(A,B))||| \leq |||\beta(f(A), f(B))||| \tag{2.1}$$

holds for all positive matrices A, B. If $\alpha = \sharp_t$, then

$$\left|\left|\left|f\left(\exp\left(\frac{A+B}{2}\right)\right)\right|\right|\right| \leq |||\beta(f(\exp A), f(\exp B))||| \tag{2.2}$$

holds.

Theorem 2.1 enables us to examine many kinds of function rather that the classical convex functions. Following, we give some examples.

It is known that the function $x \mapsto \exp(x)$ and $x \mapsto x^r$ $(r < 0)$ are ∇-$\sharp$-convex, see [3, 5]. Theorem 2.1 then gives:

Corollary 2.2. *If $r < 0$, then*

$$|||(A+B)^r||| \leq 2^r\, |||A^r \sharp B^r||| \tag{2.3}$$

and

$$|||exp(1/2(A+B))||| \leq ||| \exp(A) \sharp \exp(B)||| \tag{2.4}$$

holds for all positive matrices A, B.

The function $f(x) = x^r$ is !-!-convex on $(0,\infty)$ for every $r \in [0,1]$. Applying Theorem 2.1 we have the next result.

Corollary 2.3. *If $r \in [0,1]$, then*

$$\left|\left|\left|\left(A^{-1}+B^{-1}\right)^{-r}\right|\right|\right| \leq 2^{1-r} \left|\left|\left|\left(A^{-r}+B^{-r}\right)^{-1}\right|\right|\right| \tag{2.5}$$

holds for all positive matrices A, B.

If $r \geq 0$ of $r \leq -1$, then $x \mapsto \exp(x^r)$ is a !-$\sharp$-convex function on $(0,\infty)$. Consequently, Theorem 2.1 implies that:

Corollary 2.4. *$f\, r \geq 0$ of $r \leq -1$, then*

$$\left|\left|\left|\exp\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-r}\right|\right|\right| \leq |||\exp(A^r)\sharp\exp(B^r)||| \tag{2.6}$$

holds for all positive matrices A, B.

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Oral Presentation **$*$: Speaker**

GENERALIZED FJ AND KKT CONDITIONS IN NONSMOOTH NONCONVEX OPTIMIZATION

JAVAD KOUSHKI[1,*], MAJID SOLEIMANI-DAMANEH[2]

[1] *Department of Applied Mathematics, Faculty of Mathematics, K. N. Toosi University of Technology, Tehran, Iran*
j.koushki@kntu.ac.ir

[2] *School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran*
m.soleimani.d@ut.ac.ir

Abstract. In this talk, we investigate optimality conditions for nonsmooth nonconvex optimization problems by means of generalized Fritz John (FJ) and Karush-Kuhn-Tucker (KKT) conditions. We obtain alternative-type optimality conditions, which could be helpful in analyzing duality results and sketching numerical algorithms.

1. Introduction

FJ and KKT conditions play a central role in optimization (both theoretically and numerically). Many researchers have examined these conditions under different assumptions. Consider the following optimization problem with inequality constraints and a nonempty geometric constraint set $X \subseteq \mathbb{R}^n$:

$$\begin{array}{ll} \min & f(x) \\ s.t. & g_i(x) \leq 0,\ i = 1, 2, \cdots, m, \\ & x \in X, \end{array} \tag{1.1}$$

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[*] Speaker.

in which $f, g_i : \mathbb{R}^n \longrightarrow \mathbb{R},\ i = 1, 2, \ldots, m$, are real-valued functions. We show the feasible solutions set of (1.1) by

$$F := \{x \in X : g_i(x) \leq 0,\ i = 1, 2, \cdots, m\},$$

and the set of indices of the active constraints at $\bar{x} \in F$ by

$$I(\bar{x}) := \{i \in \{1, 2, \cdots m\} : g_i(\bar{x}) = 0\}.$$

For a set $K \subseteq \mathbb{R}^n$, the nonnegative polar cone of K, the tangent cone of K at $y \in clK$, and the normal cone of K at $y \in clK$, denoted by K°, $T_K(y)$, and $N_K(y)$, respectively, and defined as

$$K^\circ := \{z \in \mathbb{R}^n : z^T y \geq 0,\ \forall y \in K\},$$

$$T_K(y) := \Big\{ d \in \mathbb{R}^n : \exists \Big(t_\nu > 0,\ \{y^\nu\} \subseteq K \Big)\ s.t.\ \ y^\nu \longrightarrow y,\ t_\nu(y^\nu - y) \to d \Big\},$$

$$N_K(y) = -[T_K(y)]^\circ.$$

If the functions appeared in problem (1.1) are differentiable, $\bar{x} \in F$ is said to be an FJ point of (1.1) if there are non-negative coefficients $\lambda_0, \lambda_i \geq 0,\ i \in I(\bar{x})$, not all zero, such that

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) \in [T_X(\bar{x})]^\circ. \tag{1.2}$$

If $\bar{x} \in intX$, then $T_X(\bar{x}) = \mathbb{R}^n$ and $[T_X(\bar{x})]^\circ = \{0_n\}$. In this case, the above-mentioned FJ condition is reduced to the well-known classic form. Also, if $\lambda_0 \neq 0$, then we reach the KKT condition.

In the following, we present Flores-Bazan and Mastroeni's definition [1] of the FJ and KKT points, which takes into account any arbitrary set $B \subseteq \mathbb{R}^n$ instead of the tangent cone.

Definition 1.1. [1] Let $B \subseteq \mathbb{R}^n$ be a given nonempty set. Assuming differentiability of f and g_i's, a vector $\bar{x} \in F$ is called a

(i) B-FJ point of (1.1) if there exist scalars $\lambda_0, \lambda_i \geq 0,\ i \in I(\bar{x})$, not all zero, satisfying

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) \in B^\circ.$$

(ii) B-KKT point of (1.1) if there exist scalars $\lambda_i \geq 0,\ i \in I(\bar{x})$, satisfying

$$\nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) \in B^\circ.$$

2. Alternative-type FJ and KKT optimality conditions

Let $Z \subseteq \mathbb{R}^n$. The interior, the closure, the relative interior, and the boundary of Z are denoted by $int\, Z$, $cl\, Z$, $ri\, Z$, and $bd\, Z$, respectively. The convex hull and the cone generated by Z are denoted by $conv\, Z$, and $cone\, Z$, respectively. Recall that $cone\, Z := \bigcup_{t \geq 0} tZ$.

Definition 2.1. [3] Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be locally Lipschitz at $\bar{x} \in \mathbb{R}^n$. The generalized directional derivative of f at $\bar{x} \in \mathbb{R}^n$ in direction $d \in \mathbb{R}^n$ is defined by

$$f^\circ(\bar{x}; d) := \limsup_{\substack{y \longrightarrow \bar{x} \\ t \downarrow 0}} \frac{f(y + td) - f(y)}{t}.$$

Moreover, the Clarke subdifferential (generalized gradient) of f at $\bar{x} \in \mathbb{R}^n$ is the set

$$\partial f(\bar{x}) := \{\xi \in \mathbb{R}^n : f^\circ(\bar{x}; d) \geq \xi^T d, \quad \forall d \in \mathbb{R}^n\}.$$

Theorem 2.2. [3, Theorem 5.1.6] *Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ and attains its local minimum over the set $C \subseteq \mathbb{R}^n$ at $\bar{x}$. Then*

$$0 \in \partial f(\bar{x}) + N_C(\bar{x}).$$

Let $B \subseteq \mathbb{R}^n$ be a given nonempty set. Consider the following sublinear problem corresponding to Problem (1.1) and given B:

$$\begin{array}{rll} \mu := & \inf & f^\circ(\bar{x}; d) \\ & s.t. & d \in G_o(\bar{x}), \end{array} \tag{2.1}$$

where $G_o(\bar{x}) := \{d \in cl\, conv B : g_i^\circ(\bar{x}; d) < 0, \ \forall i \in I(\bar{x})\}$. Set $\mu := +\infty$ whenever $G_o(\bar{x}) = \emptyset$.

Definition 2.3. [2] Suppose that $f, g_i, \ i \in I(\bar{x})$, are locally Lipschitz at $\bar{x} \in F$. The vector $\bar{x}$ is called a

(i) FJ point of (1.1) if there exist $\lambda_0, \lambda_i \geq 0, \ i \in I(\bar{x})$, not all zero, $\bar{\xi} \in \partial f(\bar{x})$, and $\bar{\zeta}_i \in \partial g_i(\bar{x}), \ i \in I(\bar{x})$, such that

$$\lambda_0 \bar{\xi} + \sum_{i \in I(\bar{x})} \lambda_i \bar{\zeta}_i \in B^\circ. \tag{2.2}$$

(ii) KKT point of (1.1) if there exist $\bar{\xi} \in \partial f(\bar{x}), \ \bar{\zeta}_i \in \partial g_i(\bar{x}), \ \lambda_i \geq 0;$ $i \in I(\bar{x})$, such that

$$\bar{\xi} + \sum_{i \in I(\bar{x})} \lambda_i \bar{\zeta}_i \in B^\circ. \tag{2.3}$$

The next results have been reported in our recent work, [2]. Based on the following Theorem, we can derive an alternative-type FJ optimality condition.

Theorem 2.4. *Suppose that $\bar{x} \in X$ and $f, g_i, \ i \in I(\bar{x})$, are locally Lipschitz at $\bar{x}$. Then one and only one of the following two statements is true.*

(i) *There exists $d \in cl\, conv B$ such that*

$$\begin{array}{l} f^\circ(\bar{x}; d) < 0, \\ g_i^\circ(\bar{x}; d) < 0, \quad i \in I(\bar{x}). \end{array} \tag{2.4}$$

(ii) *$\bar{x}$ is a FJ point of (1.1).*

Furthermore, if B is a cone, then

$$(i) \Longleftrightarrow \mu = -\infty.$$

In consequence of Theorem 2.4, Corollary 2.5 provides an FJ necessary optimality condition.

Corollary 2.5. *Assume that $\bar{x} \in F$ and f, g_i, $t = 1, 2, \cdots, m$, are locally Lipschitz at $\bar{x}$. Furthermore, suppose that $cl\, convB \subseteq T_X(\bar{x})$. Then the FJ condition (2.2) is satisfied if x is a local optimal solution to (1.1).*

Corollary 2.6 provides a KKT necessary optimality conditions under some assumptions. The first assumption is related to the set B, and second one is corresponding to the feasibility of the sublinear Problem (2.1).

Corollary 2.6. *Let $\bar{x} \in F$ be given. Assume that f, g_i, $t = 1, 2, \cdots, m$, are locally Lipschitz at $\bar{x}$. Furthermore, assume that $cl\, convB \subseteq T_X(\bar{x})$. If $\bar{x}$ is a local optimal solution to (1.1), then under either (a) or (b) $\bar{x}$ is a KKT point of (1.1).*

(a) $conv\{\xi_i : \xi_i \in \partial g_i(\bar{x}),\ i \in I(\bar{x})\} \cap B^\circ = \emptyset$;
(b) *There exists some $d \in cl\, convB$ such that $g_i^\circ(\bar{x}; d) < 0$, for any $i \in I(\bar{x})$.*

Theorem 2.7 provides a necessary and sufficient condition equivalent to KKT conditions. Given $\Omega \subseteq \mathbb{R}^n$, define

$$F(\Omega) := \left\{ \begin{pmatrix} f^\circ(\bar{x}; d) \\ g^\circ_{I(\bar{x})}(\bar{x}; d) \end{pmatrix} : d \in \Omega \right\}.$$

in which, $g^\circ_{I(\bar{x})}(\bar{x}; d)$ is a $|I(\bar{x})|$-vector whose components are $g_i^\circ(\bar{x}; d)$, $i \in I(\bar{x})$.

Theorem 2.7. *Let $\bar{x} \in F$ and $B \subseteq \mathbb{R}^n$ be a nonempty cone. Assume that f, g_i, $i = 1, 2, \cdots, m$, are locally Lipschitz at $\bar{x}$. Then $\bar{x}$ is a KKT point for (1.1) if and only if*

$$cl\Big[F(cl\, convB) + \big(\mathbb{R}_+ \times \mathbb{R}_+^{I(\bar{x})}\big)\Big] \bigcap -\big(\mathbb{R}_{++} \times \{0\}\big) = \emptyset.$$

More results will be presented in the related talk.

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Oral Presentation

ON $\mathcal{MT}$-CYCLIC CONTRACTIONS

HOSEIN LAKZIAN

Department of Mathematics, Payame Noor University, P.O. Box 19395-4697, Tehran, I.R. of Iran
h_lakzian@pnu.ac.ir

Abstract. In this paper, we introduced the notion Hardy-Rogers $\mathcal{MT}$-cyclic contraction. Using this concept, we investigate the existence of best proximity point for such mappings in metric spaces. The uniqueness of this point will be obtain by imposing an additional condition, so called "property UC". At the end, using the definition of $\mathcal{MT}$-cyclic orbital contraction, we shall prove and discuss the existence and uniqueness of fixed point of such mappings in the setting of metric space and b-metric space.

1. Introduction

In the last decades, both fixed point theory and best proximity point theory have been appreciated by several authors, see e.g. [1-6]. In this paper, we examine existence and uniqueness of best proximity points and fixed points for generalized $\mathcal{MT}$- cyclic contractions and $\mathcal{MT}$-cyclic orbital contractions with respect to φ in the context of metric space. Let A and B be nonempty subsets of metric space (X, d). A map $T : A \cup B \to A \cup B$ is called a *cyclic* if $T(A) \subset B$ and $T(B) \subset A$, see e.g. [6] and [12]. For any nonempty subsets A and B of X, we let

$$dist(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

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A point $x \in A \cup B$ is called to be a best proximity point for T if $d(x, Tx) = dist(A, B)$. Note that if $A = B$ then the best proximity point of T turns into fixed point of T.

The concept of *property UC* was introduced by Suzuki *et al.* [14] as follows:

A pair (A, B) is said to satisfy the property UC if the following holds:

(UC) If $\{(x_n)_{n=1}^{\infty}\}$ and $\{(x'_n)_{n=1}^{\infty}\}$ are sequences in A and $\{y_n\}_{n=1}^{\infty}$ is a sequence in B such that $\lim_{n\to\infty} d(x_n, y_n) = d(A, B)$ and $\lim_{n\to\infty} d(x'_n, y_n) = d(A, B)$ then $\lim_{n\to\infty} d(x_n, x'_n) = 0$.

Definition 1.1. [8] A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $\mathcal{MT}$-*function* if it satisfies Mizoguchi-Takahashi's condition (i.e. $\limsup_{s\to t^+} \varphi(s) < 1$ for all $t \in [0, \infty)$).

Remark 1.2. [8] It is obvious that if $\varphi : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class. But it is worth to mention that there exist functions which are not $\mathcal{MT}$-functions.

Example 1.3. [8] Let $\varphi : [0, \infty) \to [0, 1)$ be defined by

$$\varphi(t) := \begin{cases} \frac{\sin t}{t} & , \text{if } t \in (0, \frac{\pi}{2}] \\ 0 & , \text{otherwise.} \end{cases}$$

Since $\limsup_{s\to 0^+} \varphi(s) = 1$, φ is not an $\mathcal{MT}$-function.

The aim of this paper is generalization of Theorem 1 in [10] by applying the notion of $\mathcal{MT}$-cyclic contraction with respect to a $\mathcal{MT}$-function φ.For convenience of the reader, we recall some of $\mathcal{MT}$-cyclic contractions in the framework of complete metric spaces which are defined in some papers:
For mapping $T : A \cup B \to A \cup B$ with $T(A) \subset B$ and $T(B) \subset A$; T is called [6] [$\mathcal{MT}$-*cyclic contraction*] if

$$d(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + (1 - \varphi(d(x, y)))dist(A, B);$$

[13][$\mathcal{MT}$-*cyclic Kannan contraction*] if

$$d(Tx, Ty) \leq \frac{1}{2}\varphi(d(x, y))\Big(d(x, Tx) + d(y, Ty)\Big) + (1 - \varphi(d(x, y)))dist(A, B);$$

[4][$\mathcal{MT}$-*cyclic Reich contraction*] if

$$d(Tx, Ty) \leq \frac{1}{3}\varphi(d(x, y))\Big(d(x, y)+d(x, Tx)+d(y, Ty)\Big)+(1-\varphi(d(x, y)))dist(A, B);$$

[2][*generalized* $\mathcal{MT}$-*cyclic contraction*] if

$$d(Tx, Ty) \leq \varphi(d(x, y)) \max\{d(x, y), d(x, Tx), d(y, Ty)\}+(1-\varphi(d(x, y)))dist(A, B).$$

It is showed there exists an example give a map T which is a $\mathcal{MT}$-cyclic contraction but not a cyclic contraction; see Example A in [6].

2. Best Proximity point for Hardy-Rogers $\mathcal{MT}$-cyclic contraction

In this section, we present our main results. We, first, introduce the *generalized $\mathcal{MT}$-cyclic contraction* with respect to auxiliary $\mathcal{MT}$-function φ.

Definition 2.1. Let A and B be nonempty subsets of a metric space (X, d). If a map $T : A \cup B \to A \cup B$ satisfies

(HRMT1) $T(A) \subset B$ and $T(B) \subset A$;
(HRMT2) there exists a $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that

$$\begin{aligned} d(Tx, Ty) \leq & \frac{\varphi(d(x, y))}{5}\Big(d(x, y) + d(x, Tx) + d(Ty, y) + d(x, Ty) + d(Tx, y)\Big) \\ & + \big(1 - \varphi(d(x, y))\big) dist(A, B), \end{aligned}$$

for all $x \in A$ and $y \in B$. Then T is called a *Hardy-Rogers $\mathcal{MT}$-cyclic contraction with respect to φ* on $A \cup B$.

In what follows that we establish the following theorem for best proximity point which is one of the main results in this paper.

Theorem 2.2. *Let A and B be nonempty subsets of a metric space (X, d) and (A, B) satisfies the property UC. Let $T : A \cup B \to A \cup B$ be a cyclic map and let φ be a $\mathcal{MT}$-function. Suppose that A is complete and T is a Hardy-Rogers $\mathcal{MT}$-cyclic contraction with respect to φ. Then the following hold:*

(i) *T has a best proximity point z in A.*
(ii) *z is a unique fixed point of T^2 in A.*
(iii) *$\{T^{2n}x\}$ converges to z for every $x \in A$.*
(iv) *T has at least one best proximity point in B.*
(v) *If (B, A) satisfies the property UC, then Tz is unique best proximity point in B and $\{T^{2n}y\}$ converges to Tz for every $y \in B$.*

3. Best Proximity point for $\mathcal{MT}$-cyclic orbital contraction in b-metric spaces

In this section we obtain fixed point theorem for $\mathcal{MT}$-cyclic orbital contraction in b-metric spaces.

Bakhtin [3] and Czerwik [5] introduced b-metric spaces (a generalization of metric spaces) and proved the contraction principle in this framework.

Definition 3.1. [3] and [5] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A triplet (X, d, s), is called a b-metric space with coefficient s.

We first introduce the concept of MT-cyclic orbital contractions.

Definition 3.2. Let A and B be nonempty subsets of a metric space (X, d). If a map $T : A \cup B \to A \cup B$ satisfies

(MTO1) $T(A) \subset B$ and $T(B) \subset A$;

(MTO2) for some $x \in A$ there exists a MT-function $\varphi_x : [0, \infty) \to [0, 1)$ such that

$$d(T^{2n}x, Ty) \leq \varphi_x(d(T^{2n-1}x, y))d(T^{2n-1}x, y) \quad \text{for any } y \in A \text{ and } n \in \mathbb{N}.$$

Then T is called a *$\mathcal{MT}$-cyclic orbital contraction with respect to φ_x* on $A \cup B$.

The following example give a map T which is a $\mathcal{MT}$-cyclic orbital contraction but not a cyclic orbital contraction.

We obtain a unique fixed point for such a map as follows.

Theorem 3.3. *Let A and B be nonempty closed subsets of a complete b-metric space (X, d, s) and $T : A \cup B \to A \cup B$ be a $\mathcal{MT}$-cyclic orbital contraction with respect to φ. Then $A \cap B$ is nonempty and T has a unique fixed point.*

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MAJORIZATION AND STOCHASTIC LINEAR MAPS IN VON NEUMANN ALGEBRAS

SEYED MAHMOUD MANJEGANI

Department of Mathematical Science
Isfahan University of Technology
Isfahan, Iran
manjgani@iut.ac.ir

Abstract. The aim of the present paper is to introduce semi-doubly stochastic and (weak)majorization on a non commutative measure space $(\mathcal{M}, \tau)$, where $\mathcal{M}$ is a semi finite von Neumann algebra with a normal faithful trace τ.

1. Introduction

Since Hardy, Litttewood, and Pólya in 1929 introduced the concept of majorization, many mathematicians have discussed the weak majorization and manjorization in various circumstances with several applications. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$. x is said to be majorized and denoted by y $x \prec y$, if $\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}$, for all $1 \leq k \leq n$ and $\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow}$,where $x^{\downarrow} = (x_1^{\downarrow}, \ldots, x_n^{\downarrow})$ and $y^{\downarrow} = (y_1^{\downarrow}, \ldots, y_n^{\downarrow})$ are obtained from x and y by rearranging their components in decreasing order. Moreover, the study of (weak)majorization has been successful in the theory of matrices via comparison of eigenvalues by Ando in 1982. On the other hand, the doubly stochastic matrices and maps have been studied in connection with majorization theory by Mirsky, Chong, Alberti and Uhlman.

Definition 1.1. An $n \times n$ matrix $D = (a_{ij})$ is called doubly stochastic if $D\mathbf{1} = \mathbf{1}$ and $D^*\mathbf{1} = \mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$ and D^* is the adjoint matrix of D.

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Theorem 1.2. [5] For $X, Y \in \mathbb{R}^n$, the following statements are equivalent:

(1) $X \prec Y$.
(2) X is in the convex hull of all vectors obtained by permuting the coordinates of Y.
(3) $X = DY$ for some doubly stochastic matrix D.

Definition 1.3. Let A and B are two $m \times n$ matrices. A is majorized by B in symbols $A \prec B$ if there is a doubly stochastic $m \times m$ matrix D such that $A = DB$.

The theory of (weak)majorization has been developed for real- valued measurable functions on abstract measure space based on the theory rearrangements by Chong and Sakai. In the case of a σ-finite measure space $(X, \mathcal{A}, \mu)$, the notion of decreasing rearrangement can be defined for non-negative integrable functions. For a finite measure space $(X, \mathcal{A}, \mu)$, Ryff considered the class of all linear operators $T : L^1(X, \mu) \to L^1(X, \mu)$ for which $Tf \prec f$, for all $f \in L^1(X, \mu)$. This class is denoted by $\mathcal{DS}(L^1(X, \mu))$ and each element of this class is called doubly stochastic operators. For σ-finite measure space $(X, \mathcal{A}, \mu)$, in [1] the semi-doubly stochastic operator is introduced and the set of all these operators is denoted by $\mathcal{SDS}\big(L^1(X, \mu)\big)$. For a non-negative $f \in L^1(X, \mu)$, let $\mathcal{S}_f := \{Sf; S \in \mathcal{SDS}(L^1)\}$ and $\Omega_f := \{h \in L^1; h \geq 0 \text{ and } h \prec f\}$. It is easily seen that both sets $\mathcal{S}_f$ and Ω_f are convex subsets of L^1. It has been proved that $\mathcal{S}_f$ is dense in Ω_f [1].

2. Main results

In this section we study the relation between majorization and doubly stochastic maps on a semi finite von Neumann algebras. Throughout this section $\mathcal{M}$ is a semi finite von Neumann algebra on a Hilbert space $\mathcal{H}$ and τ is a faithful normal semi finite trace on $\mathcal{M}$. We fix a couple $(\mathcal{M}, \tau)$ as a non-commutative measure space. For positive operator x affiliated with We fix a couple $(\mathcal{M}, \tau)$, $e_I(x)$ will denote the spectral projection of x corresponding to an interval I in $[0, \infty)$. A closed and densely defined linear operator $x : \mathcal{D}(x) \to \mathcal{H}$ is said to be τ-measurable if x affiliated with $\mathcal{M}$, and there exists $\lambda \geq 0$ such that $\tau(e^{|x|}(\lambda, \infty)) < \infty$. The collection of all τ-measurable operators is denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a complex $*$-algebra with unit element 1. The von Neumann algebra $\mathcal{M}$ is a $*$-subalgebra of $L_0(\mathcal{M})$. For each $\mathfrak{L}$ of $L_0(\mathcal{M})$, the set of all positive elements in $\mathfrak{L}$ is denoted by $\mathfrak{L}_+$. The closure of $L_1(\mathcal{M})$ in $L_0(\mathcal{M})$ is denoted by $\widetilde{\mathtt{G}}$.

Let $x \in L_0(\mathcal{M})$ and $t > 0$. The t-th singular value of x (or generalized s-numbers) is the number denoted by $\mu_t(x)$ and for each $t \in \mathbb{R}_0^+$ is defined by

$$\mu_t(x) = \inf\{\| xe \|: e \in \mathcal{P}(\mathcal{M}), \tau(1 - e) \leq t\}.$$

For $0 < p < \infty$, $L_p(\mathcal{M},\tau)$ is defined as the set of all τ-measurable operators x such that

$$\| x \|_p = \tau(|x|^p)^{\frac{1}{p}} < \infty. \tag{2.1}$$

Moreover, we put $L_\infty(\mathcal{M},\tau) = \mathcal{M}$ and denote by $\| \cdot \|_\infty$ the usual operator norm. For simplicity from now on $L_p(\mathcal{M},\tau)$ will denoted by $L_p(\mathcal{M})$. Let $1 \le p < \infty$, an operator $x \in \mathcal{M}$ is said to be locally integrable if there exists $\delta > 0$ such that

$$\int_0^\delta \mu_t(x)^p \, dt < \infty.$$

The set containing all these operators is denoted by $\mathfrak{L}^p_{loc}(\mathcal{M})$. Note that in particular, all bounded operators $a \in \mathcal{M}$ are of this class. Moreover,

$$\int_0^\delta \mu_t(x)^p \, dt \ge \mu_\delta(x)^{p-1} \int_0^\delta \mu_t(x) \, dt$$

implies that $\mathfrak{L}^p_{loc}(\mathcal{M}) \subset \mathfrak{L}^1_{loc}(\mathcal{M})$ for each $p \ge 1$ [3].

Definition 2.1. Let a, b be positive τ-measureable operators. We say that a is submajorized (weakly majorized) by b in symbol $a \prec_w b$, if $\int_0^s \mu_t(a)dt \le \int_0^s \mu_t(b)dt$ for all $s > 0$. Moreover a is said to be majorized by b and is indicated by $a \prec b$, if $a \prec_w b$ and $\int_0^\infty \mu_t(a)dt = \int_0^\infty \mu_t(b)dt$.

Let φ be a linear map from $\mathcal{M}$ to itself. φ is positive if $\varphi(a)$ is positive for every $a \in \mathcal{M}_+$, φ is unital if $\varphi(1) = 1$ and φ is trace preserving if $\tau(\varphi(a)) = \tau(a)$.

Definition 2.2. [2] A positive linear map $\varphi : \mathcal{M} \longrightarrow \mathcal{M}$ is called doubly stochastic if it is unital and trace preserving. φ is called doubly substochastic if $\varphi(1) \le 1$ and $\tau(\varphi(a)) \le \tau(a)$ for all $a \in \mathcal{M}_+$. The set of all doubly stochastic (resp. doubly substochastic) linear maps on $\mathcal{M}$ is denoted by $\mathcal{DS}(\mathcal{M})$(resp. $\mathcal{DSS}(\mathcal{M})$).

In the following two propositions, which are proved in [2], the relations between (weak)majorization and doubly (sub)stochastic maps are investigated.

Proposition 2.3. Let $\varphi : \mathcal{M} \longrightarrow \mathcal{M}$ be a positive linear map. Then

(1) $\varphi(a) \prec_w a$ for all $a \in \mathcal{M}_+$ if and only if $\varphi \in \mathcal{DSS}(\mathcal{M})$.
(2) $\varphi(a) \prec a$ for all $a \in \mathcal{M}_+$ if and only if $\varphi \in \mathcal{DSS}(\mathcal{M})$ and φ is trace preserving (hence $\varphi \in \mathcal{DS}(\mathcal{M})$ when $\tau(1) < \infty$).

Proposition 2.4. Let $a, b \in L_0(\mathcal{M})$.

(1) If $\tau(1) < \infty$ and $b \in L_1(\mathcal{M})$, then $a \prec b$ if and only if there exists $\varphi \in \mathcal{DS}(\mathcal{M})$ such that $a = \varphi(b)$.
(2) If $b \in L_p(\mathcal{M})$ with $1 \le p < \infty$, or if $b \in \mathcal{M}$ and $a \in \widetilde{\mathtt{G}}$, then $a \prec_w b$ if and only if $\varphi \in \mathcal{DSS}(\mathcal{M})$ such that $a = \varphi(b)$.

Moreover, a normal and completly positive φ can be chosen in each (1) and (2) if $b \in L_p(\mathcal{M})$, $1 \le p < \infty$, or $a \in \mathtt{G} = \widetilde{\mathtt{G}} \cap \mathcal{M}$.

Theorem 2.1. [4] *Let x, y be operators in $L_0(\mathcal{M})$. Then for $p, q, r \in \mathbb{R}^+$ that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$,*

$$\frac{1}{r} \mid xy^* \mid^r \prec_w \frac{1}{p} \mid x \mid^p + \frac{1}{q} \mid y \mid^q \ . \tag{2.2}$$

Moreover, if $x, y \in L_1(\mathcal{M})$ are bounded operators or $xy \in \mathfrak{L}^2_{loc}(\mathcal{M})$, then there exists a $\varphi \in \mathcal{DS}(\mathcal{M})$ such that $a = \varphi(b)$.

$$\frac{1}{r} \mid xy^* \mid^r \prec \frac{1}{p} \mid x \mid^p + \frac{1}{q} \mid y \mid^q , \tag{2.3}$$

if and only if $\mid x \mid^p = \mid y \mid^q$.

Corollary 2.5. Let $x, y \in L_1(\mathcal{M})$ are bounded operators. Then for $p, q, r \in \mathbb{R}^+$ that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, there exists $\varphi \in \mathcal{DSS}(\mathcal{M})$ such that

$$\mid xy^* \mid^r = \varphi\left(\frac{r}{p} \mid x \mid^p + \frac{r}{q} \mid y \mid^q\right).$$

Moreover, if $\tau(1) < \infty$, then $\varphi \in \mathcal{DS}(\mathcal{M})$

3. Conclusion

For $a \in L_0(\mathcal{M})_+$, let $\mathcal{S}_a := \{\varphi(x);\ \varphi \in \mathcal{DSS}(\mathcal{M})\}$, $\mathcal{D}_a := \{\varphi(x);\ \varphi \in \mathcal{DS}(\mathcal{M})\}$ and $\Omega_a := \{b \in L_0(\mathcal{M})_+;\ b \prec a\}$. Sets $\mathcal{S}_a$, $\mathcal{D}_a$ and Ω_a are convex. Proposition 2.4 (part (1)) implies that If $\tau(1) < \infty$ and $a \in L_1(\mathcal{M})$, then $\mathcal{S}_a = \mathcal{D}_a$. When $\tau(1) = \infty$, it is not clear for us whether or not $\mathcal{S}_a = \Omega_a$. If we consider $(\Omega\mathcal{S})_a := \{b \in L_0(\mathcal{M})_+;\ b \prec_w a\}$, then Proposition 2.4 (part (2)) implies that $\mathcal{S}_a = (\Omega\mathcal{S})_a$ when $a \in L_1(\mathcal{M})$.

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Oral Presentation

$F(\psi, \varphi)$-CONTRACTIONS ON M-METRIC SPACES

FAHIMEH MIRDAMADI

Department of Mathematics, Islamic Azad University, Isfahan Branch, Isfahan, Iran
mirdamadi.f@gmail.com

Abstract. Partial metric spaces were introduced by Matthews in 1994 as a part of the study of denotational semantics of data flow networks. In 2014 Asadi and *et al.* [1] extend the Partial metric spaces to M-metric spaces. In this work, we introduce the class of $F(\psi, \varphi)$-contractions and investigate the existence and uniqueness of fixed points for the new class $\mathcal{C}$ in the setting of M-metric spaces. The theorems that we prove generalize many previously obtained results. We also give some examples showing that our theorems are indeed proper extensions.

1. Introduction

The notion of metric space was introduced by Fréchet [2] in 1906. Later, many authors attempted to generalize the notion of metric space such as pseudo metric space, quasi metric space, semi metric spaces and partial metric spaces. In this paper, we consider another generalization of a metric space, so called M-metric space. This notion was introduced by Asadi *et al.* (see e.g. [1]) to solve some difficulties in domain theory of computer science. Geraghty in 1973 introduced an interesting class of auxiliary function to refine the Banach contraction mapping principle. Let $\mathcal{F}$ denote all functions $\beta : [0, \infty) \to [0, 1)$ which satisfies the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

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By using the function $\beta \in \mathcal{F}$ Geraghty [3] proved the following remarkable theorem.

Theorem 1.1. (Geraghty [3]) *Let (X, d) be a complete metric space and $T : X \to X$ be an operator. Suppose that there exists $\beta : [0, \infty) \to [0, 1)$ satisfying the condition,*

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0$$

If T satisfies the following inequality

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y), \text{ for any } x, y \in X, \tag{1.1}$$

then T has a unique fixed point.

In 2014 Asadi *et al.* [1] introduced the M-metric space which extends partial metric space [4], by some of certain fixed point theorems obtained therein, and they have given a theorem that its proof is still open as follows.

Theorem 1.2. *Let (X, m) be a complete M-metric space and $T : X \to X$ be mapping satisfying:*

$$\exists k \in [0, \frac{1}{2}) \quad \text{such that} \quad m(Tx, Ty) \leq k(m(x, Ty) + m(y, Tx)) \quad \forall x, y \in X.$$

Admissible mappings have been defined recently by Samet et al [**?**] and employed quite often in order to generalize the results on various contractions. We state next the definitions of α-admissible mapping and triangular α-admissible mappings.

Definition 1.3. Let $\alpha : X \times X \to [0.\infty)$. A self-mapping $T : X \to X$ is called α-admissible if the condition

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1, \tag{1.2}$$

is satisfied for all $x, y \in X$.

Definition 1.4. A mapping $T : X \to X$ is called triangular α-admissible if it is α-admissible and satisfies

$$\alpha(x, y) \geq 1, \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1. \tag{1.3}$$

where $x, y, z \in X$ and $\alpha : X \times X \to [0.\infty)$ is a given function.

In what follows we recall the notion of (triangular) α-orbital admissible, introduced by Popescu [6], that is inspired from [5].

Definition 1.5. [6] For a fixed mapping $\alpha : X \times X \to [0, \infty)$, we say that a self-mapping $T : X \to X$ is an α-orbital admissible if

$$(O1) \;\; \alpha(u, Tu) \geq 1 \Rightarrow \alpha(Tu, T^2u) \geq 1.$$

Let $\mathcal{A}$ be the collection of all α-orbital admissible $T : X \to X$.

In addition, T is called triangular α-orbital admissible if T is α-orbital admissible and

$$(O2) \;\; \alpha(u, v) \geq 1 \text{ and } \alpha(v, Tv) \geq 1 \Rightarrow \alpha(u, Tv) \geq 1$$

Let $\mathcal{O}$ be the collection of all triangular α-orbital admissible $T : M \to M$.

Definition 1.6. ([1]) Let X be a non empty set. A function $m : X \times X \to \mathbb{R}^+$ is called M-metric if the following conditions are satisfied:

(m1) $m(x,x) = m(y,y) = m(x,y) \iff x = y$,
(m2) $m_{xy} \le m(x,y)$,
(m3) $m(x,y) = m(y,x)$,
(m4) $(m(x,y) - m_{xy}) \le (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$.

Where

$$m_{xy} := \min\{m(x,x), m(y,y)\} = m(x,x) \vee m(y,y),$$

Then the pair (X,m) is called a M-metric space.

Definition 1.7. A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Remark 1.8. We let Ψ denote the class of the altering distance functions.

Definition 1.9. An ultra altering distance function is a continuous, non-decreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) \ge 0$.

Remark 1.10. We let Φ denote the class of the ultra altering distance functions.

Definition 1.11. A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called C-*class* function if it is continuous and satisfies following axioms:

(1) $F(s,t) \le s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$; for all $s,t \in [0,\infty)$.

Note for some F we have that $F(0,0) = 0$.
We denote C-class functions as $\mathcal{C}$.

2. MAIN RESULTS

Definition 2.1. Let (X,m) be an M-metric space, and let $T : X \to X$ be a given mapping. We say that T is $F(\alpha,\psi)$-contractive mapping if there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$ such that

$$\psi(m(Tx,Ty)) \le F(\psi(m(x,y)), \varphi(m(x,y))), \tag{2.1}$$

Definition 2.2. Let (X,m) be an M-metric space, and let $T : X \to X$ be an α-admissible mapping. We say that T is an α-admissible $F(\alpha,\psi)$-contractive mapping if there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$ such that

$$\alpha(x,y)\psi(m(Tx,Ty)) \le F(\psi(m(x,y)), \varphi(m(x,y))), \tag{2.2}$$

Definition 2.3. Let (X,m) be an M-metric space, and let $T : X \to X$ be an α-admissible mapping. We say that T is a generalized α-admissible $F(\alpha,\psi)$-contractive mapping if there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$ such that

$$\alpha(x,y)\psi(m(Tx,Ty)) \le F(\psi(M(x,y)), \varphi(M(x,y))), \tag{2.3}$$

Where $M(x,y) = \max\{m(x,y), m(x,Tx), m(y,Ty)\}$

Theorem 2.4. *(X,m) be a complete M-metric space, and let $T : X \to X$ be a generalized α-admissible $F(\alpha,\psi)$-contractive mapping. and satisfies the following conditions:*
(i) $T \in \mathcal{O}$;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then T has a fixed point $v \in X$ and $\{T^n x_0\}$ converges to v.

For the uniqueness of a fixed point of a generalized α-admissible $F(\alpha,\psi)$-contractive mapping, we shall suggest the following hypothesis.
$(*)$ For all $x, y \in \mathrm{Fix}(T)$, we have $\alpha(x,y) \geq 1$.
Here, $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 2.5. *Adding condition $(*)$ to the hypotheses of Theorem 2.4, we obtain that v is the unique fixed point of T.*

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COUPLED FIXED POINT THEOREMS IN ORDERED M-METRIC SPACES

FAHIMEH MIRDAMADI

Department of Mathematics, Islamic Azad University, Isfahan Branch, Isfahan, Iran
mirdamadi.f@gmail.com

ABSTRACT. Using control functions, we improve and extend some results of coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces by Lakshmikantam and Ćirić [2] to ordered M-metric spaces.

1. INTRODUCTION

Partial metric spaces, which are generalizations of metric spaces, introduced by S. G. Mathews [3] as a part of the study of denotational semantics of data flow networks and he gave a Banach fixed point result for these spaces. After that, In 2014 Asadi *et al.* [1] introduced the M-metric space which extends p-metric space, by some of certain fixed point theorems obtained therein, many authors proved fixed point theorems on M-metric spaces, see [4]. Lakshmikantam and Ćirić in [2], gave some interesting fixed point results on ordered metric spaces, but there exist two error in that paper. In this paper, using a class of control functions, we extend main results of [2] to complete M-metric spaces. We mention two errors of [2] as well.

Definition 1.1. ([1]) Let X be a non empty set. A function $m : X \times X \to \mathbb{R}^+$ is called a m-metric if the following conditions are satisfied:

(m1) $m(x,x) = m(y,y) = m(x,y) \iff x = y$,

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Key words and phrases. Fixed point, Partial metric space, M-metric space, Coupled fixed point.

(m2) $m_{xy} \le m(x,y)$,
(m3) $m(x,y) = m(y,x)$,
(m4) $(m(x,y) - m_{xy}) \le (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$.

Where
$$m_{xy} := \min\{m(x,x), m(y,y)\}.$$
Then the pair (X,m) is called an M-metric space.

The following notation is useful in the sequel.
$$M_{xy} := \max\{m(x,x), m(y,y)\}.$$
We note that every p-metric is a m-metric. In the following example we present an example of a m-metric which is not p-metric.

Example 1.2. ([1]) Let $X = \{1,2,3\}$. Define
$$m(1,2) = m(2,1) = m(1,1) = 8$$
$$m(1,3) = m(3,1) = m(3,2) = m(2,3) = 7 \quad m(2,2) = 9 \quad m(3,3) = 5$$
so m is m-metric but m is not p-metric. Since $m(2,2) \not\le m(1,2)$ means m is not p-metric. If $D(x,y) = m(x,y) - m_{x,y}$ then $m(1,2) = m_{1,2} = 8$ but it means $D(1,2) = 0$ while $1 \ne 2$ which means D is not metric.

Definition 1.3. Let $(X, \le)$ be a partially ordered set and $f : X \times X \to X$ and $g : X \to X$. We say f has the mixed g-monotone property if for any $x, y, x_1, x_2, y_1, y_2 \in X$
$$g(x_1) \le g(x_2) \text{ implies } \quad f(x_1, y) \le f(x_2, y) \tag{1.1}$$
and
$$g(y_1) \le g(y_2) \text{ implies } \quad f(x, y_1) \le f(x, y_2) \tag{1.2}$$

Definition 1.4. An element $(x,y) \in X \times X$ is called a coupled coincidence point of mappings $f : X \times X \to X$ and $g : X \to X$ if
$$f(x,y) = g(x), \quad f(y,x) = g(y).$$

Definition 1.5. Let $f : X \times X \to X$ and $g : X \to X$. We say f and g are commutative if for all $x, y \in X$
$$g(f(x,y)) = f(g(x), g(y)).$$
For simplicity, we denote $g(x)$ by gx.

2. MAIN RESULTS

Definition 2.1. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $\psi(t) \to 0$ if and only if $t \to 0$, ψ^{-1} is nondecreasing and one to one and $\psi(a+b) \le \psi(a) + \psi(b)$ (sub-additivity) for $a, b \in \mathbb{R}^+$. We denote the set of these functions by Ψ. For example, $\psi(t) = \alpha t$ and $\psi(t) = e^{\alpha t}$ for $\alpha \ge 0$ belong to Ψ.

Theorem 2.2. *Let $(X, m, \leq)$ be a partially ordered complete M-metric space. Assume that there is a function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$, $\varphi(t) < t$ and $\lim_{r \to t^+} \varphi(r) < t$ for each $t > 0$ and also suppose $f : X \times X \to X$ and $g : X \to X$ are such that f has the mixed g-monotone property and*

$$\psi(m(f(x,y), f(u,v))) \leq \varphi(\frac{\psi(m(gx, gu)) + \psi(m(gy, gv))}{2}) \tag{2.1}$$

for all $x, y, u, v \in X$ for which $gx \leq gu$ and $gy \geq gv$ and $\psi \in \Psi$. Suppose $f(X \times X) \subseteq g(X)$, g is continuous and commute with f and also suppose either
(a) f is continuous or
(b) X has the following property:

(i) *if for a non-decreasing sequence $m^w(x_n, x) \to 0$, then*

$$gx_n \leq gx \;\; \text{for all}\, n. \tag{4}$$

(ii) *if for a non-increasing sequence $m^w(y_n, y) \to 0$, then*

$$gy \leq gy_n \;\; \text{for all}\, n. \tag{5}$$

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq f(x_0, y_0) \quad \text{and} \quad g(y_0) \geq f(y_0, x_0),$$

then f and g have a coupled coincidence point.

If $(X, \leq)$ is a partially ordered set, we can endowed $X \times X$ with a partial order as follows:

$$(x_1, y_1) \leq (x_2, y_2) \Longleftrightarrow x_1 \leq x_2 \text{ and } y_2 \leq y_1.$$

The following uniqueness theorem is a generalization of Theorem 2.2 of [5].

Theorem 2.3. *If in Theorem 2.2, we also suppose that for any (x, y) and (x^*, y^*) in $X \times X$, there is $(u, v) \in X \times X$ such that $(f(u, v), f(v, u))$ is comparable to $(f(x, y), f(y, x))$ and $(f(x^*, y^*), f(y^*, x^*))$, then there is a unique point (z, w) such that $z = gz = f(z, w), w = gw = f(w, z)$.*

Remark 2.4. It is noted that, the relation (31) in the proof of Theorem 2.2 of [2], is not correct because in general φ is not non-decreasing.

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MOUNTAIN PASS SOLUTION FOR A $p(x)$-BIHARMONIC KIRCHHOFF TYPE EQUATION

MARYAM MIRZAPOUR

*Department of Mathematics, Faculty of Mathematical Sciences, Farhangian University ,
Tehran, Iran
m.mirzapour@cfu.ac.ir*

Abstract. In this paper we deal with the existence of weak solution for a $p(x)$-Kirchhoff type problem of the following form

$$\begin{cases} -\left(a - b\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)\Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda|u|^{p(x)-2}u + g(x,u) & \text{in } \Omega,\\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Using the Mountain Pass Theoem, we establish conditions ensuring the existence result.

1. Introduction

In this paper we study the following problem

$$\begin{cases} -\left(a - b\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)\Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda|u|^{p(x)-2}u + g(x,u) & \text{in } \Omega,\\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N,\ N \geq 2$ is a bounded smooth domain with smooth boundary $\partial\Omega$, $p(x) \in C(\overline{\Omega})$, $a, b > 0$ are constants, g is a continuous function, λ is a real parameter. Suppose that the nonlinearity $g(x,t) \in C(\overline{\Omega}, \mathbb{R})$ satisfies the following assumptions:

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($\mathbf{g_1}$) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and the subcritical growth condition, i.e. there exists a constant $c \geq 0$ such that

$$|g(x,s)| \leq c_1(1 + |s|^{q(x)-1}),$$

for all $(x,t) \in \Omega \times \mathbb{R}$ where $q(x) \in C_+(\overline{\Omega})$ and $q(x) < p_k^*(x)$.

($\mathbf{g_2}$) $g(x,s) = o(|s|^{p(x)-2}s)$ as $s \to 0$ uniformly with respect to $x \in \Omega$.

($\mathbf{g_3}$) There exist $M > 0$ and $\theta \in \left(p^+, \dfrac{2(p^-)^2}{p^+}\right)$ such that $0 < \theta G(x,s) \leq sg(x,s)$, for all $|s| \geq M$ and $x \in \Omega$ where $G(x,t) = \int_0^s g(x,\tau)\, d\tau$.

We mainly consider a new Kirchhoff problem involving the $p(x)$-biharmonic operator, that is, the form with a nonlocal coefficient $(a - b\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx)$. Its background is derived from nagative Young's modulus, when the atoms are pulled apart rather than compressed together and the strain is negative. Recently, the authors in [6] first studied this kind of problem

$$\begin{cases} -\left(a - b\int_\Omega |\nabla u|^2\, dx\right) \Delta u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $2 < p < 2^* := (2N)/(N-2)$, and they obtained the existence of solutions by using the mountain pass theorem. Furthermore, some interesting results have been obtained for this kind of Kirchhoff-type problem. We refer the readers to [1, 5, 7] and the references therein.

Now, we state our main result:

Theorem 1.1. *Assume that the function $q \in C(\overline{\Omega})$ satisfies*

$$1 < p^- < p(x) < p^+ < 2p^- < q^- < q(x) < p_k^*(x) := \frac{Np(x)}{N - kp(x)}.$$

Then for any $\lambda \in \mathbb{R}$, with ($\mathbf{g_1}$)-($\mathbf{g_3}$) satisfied, problem (1.1) has a nontrivial weak solution.

2. Notations and preliminaries

Let Ω be a bounded domain of $\mathbb{R}^N$, denote $C_+(\overline{\Omega}) = \{p(x);\ p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \forall x \in \overline{\Omega}\}$, $p^+ = \max\{p(x);\ x \in \overline{\Omega}\}$, $p^- = \min\{p(x);\ x \in \overline{\Omega}\}$;, $L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R}$ measurable and $\int_\Omega |u(x)|^{p(x)}\, dx < \infty\}$, with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}$.

Proposition 2.1 (See [3])**.** *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$, i.e., $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, for all $x \in \Omega$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have $\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)} |v|_{q(x)} \leq 2|u|_{p(x)}|v|_{q(x)}$.*

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as follows: $W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \ D^\alpha u \in L^{p(x)}(\Omega), \ |\alpha| \leq k\}$, where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_N^{\alpha_N}} u$, with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$ equipped with the norm $\|u\|_{k,p(x)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x)}$, also becomes a separable and reflexive Banach space. For more details, we refer the reader to [3, 2].

Proposition 2.2 (See [3])**.** *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$. If we replace $\leq$ with $<$, the embedding is compact.*

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Note that the weak solutions of problem (1.1) are considered in the generalized Sobolev space $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ equipped with the norm $\|u\| = \inf\left\{\mu > 0 : \ \int_\Omega \left|\frac{\Delta u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}$.

Remark 2.3. According to [4], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $|\Delta \cdot|_{p(x)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $|\Delta\cdot|_{p(x)}$ are equivalent.

We consider the functional $\rho(u) = \int_\Omega |\Delta u|^{p(x)}\, dx$ and give the following fundamental proposition.

Proposition 2.4 (See [**?**])**.** *For $u \in X$ and $u_n \subset X$, we have*

(1) $\|u\| < 1(\textit{respectively} = 1; > 1) \Longleftrightarrow \rho(u) < 1(\textit{respectively} = 1; > 1)$;
(2) $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;
(3) $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;
(4) $\|u_n\| \to 0$ $(\textit{respectively} \to \infty) \Longleftrightarrow \rho(u_n) \to 0$ $(\textit{respectively} \to \infty)$.

3. Proof of Theorem 1.1

We say $u \in X$ is a weak solution of (1.1), if

$$(a - b\int_\Omega \frac{1}{p(x)})|\Delta u|^{p(x)}\, dx \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta\varphi\, dx - \lambda\int_\Omega |u|^{p(x)-2}u\varphi\, dx = \int_\Omega g(x,u)\varphi\, dx,$$

where $\varphi \in X$. The energy functional $J : X \to \mathbb{R}$ associated with problem

$$\begin{aligned} J(u) = a\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\, dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\, dx\right)^2 \\ - \lambda\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\, dx - \int_\Omega G(x,u)\, dx \end{aligned} \tag{3.1}$$

for all $u \in X$ is well defind and of class C^1 in X. Moreover, we have

$$\langle J'(u),\varphi\rangle = (a - b\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx)\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta\varphi\,dx$$
$$-\lambda\int_\Omega |u|^{p(x)-2}u\varphi\,dx - \int_\Omega g(x,u)\varphi\,dx. \tag{3.2}$$

Hence, we can observe that the critical points of J are weak solutions of problem (1.1).

Definition 3.1. Let $(X,\|\cdot\|)$ be a Banach space and $J\in C^1(X)$. We say that J satisfies the Palais-Smale condition at level c ($(PS)_c$ in short) if any sequence $\{u_n\}\subset X$ satisfying $J(u_n)\to c$ and $J'(u_n)\to 0$ in X^* as $n\to\infty$, has a convergent subsequence.

Lemma 3.2. *Assume that* $(\mathbf{g_1})$- $(\mathbf{g_3})$ *hold. Then the functional* J *satisfies the* $(PS)_c$ *condition, where* $c<\dfrac{a^2}{2b}$.

Lemma 3.3. *Assume that* g *satisfies* $(\mathbf{g_1})$- $(\mathbf{g_3})$. *Then* J *satisfies the Mountain Pass geometry, that is,*

(i) *there exists* $\rho,\alpha>0$ *such that* $J(u)\geq\alpha>0$, *for any* $u\in X$ *with* $\|u\|=\rho$.

(ii) *there exists* $e\in X$ *with* $\|e\|>\rho$ *such that* $J\ (e)<0$.

By Lemmas 3.2, 3.3 and the fact that $J(0)=0$, J satisfies the Mountain Pass Theorem. Therefor, problem (1.1) has indeed a nontrivial waek solution.

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ON A SOLUTION OF NONLOCAL $(p_1(x), p_2(x))$-BIHARMONIC PROBLEM

MARYAM MIRZAPOUR

Department of Mathematics, Faculty of Mathematical Sciences, Farhangian University , Tehran, Iran
m.mirzapour@cfu.ac.ir

Abstract. This article is concerned with the existence of weak solution for a class of elliptic Navier boundary value problem involving the $(p_1(x), p_2(x))$-biharmonic operator. By means of variational methods and the theory of variable exponent Sobolev spaces, we establish the existence of a non-trivial weak solution for problem.

1. Introduction

In recent years, a great attention has been paid to the study of various mathematical problems with variable exponent. Fourth order equations appears in many contexts. Some of these problems come from different areas of applied mathematics and physics such as Micro Electro-mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells [5].
In this work, we consider the problem

$$\begin{cases} M_1\left(\int_\Omega \frac{1}{p_1(x)}|\Delta u|^{p_1(x)}\,dx\right)\Delta^2_{p_1(x)}u+ \\ M_2\left(\int_\Omega \frac{1}{p_2(x)}|\Delta u|^{p_2(x)}\,dx\right)\Delta^2_{p_2(x)}u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N,\ N \geq 2$ is a bounded smooth domain with smooth boundary $\partial\Omega,\ N \geq 1,\ \Delta^2_{p_i(x)}u := \Delta(|\Delta u|^{p_i(x)-2}\Delta u)$, is the $p(x)$-biharmonic operator

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with ,$p_i(x) \in C(\overline{\Omega})$, $(i=1,2)$ such that $1 < p_i^- := \inf_{x\in\Omega} p_i(x) \le p_i^+ := \sup_{x\in\Omega} p_i(x) < +\infty$, $M_i : \mathbb{R}^+ \to \mathbb{R}^+$ are differentiable functions and $f(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function.
Problem (1.1) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff [6]. Throughout this paper, we make the following assumptions on the function f and the Kirchhoff functions M_1 and M_2.

- $(\mathbf{H_0})$ $\exists\, m_0, m_1 > 0$ such that $M_1(t) \ge 0$ and $M_2(t) \ge 0$ for all $t \ge 0$.
- $(\mathbf{H_1})$ $\exists\, \mu_1, \mu_2 \in (0,1)$ such that $\widehat{M_i}(t) \ge (1-\mu_i)M_i(t)t$ for all $t \ge 0$, where $\widehat{M_i}(t) = \int_0^t M_i(s)\,ds, i=1,2$.
- $(\mathbf{H_2})$ M_1, M_2 are differentiable and decreasing functions on $\mathbb{R}^+$.
- $(\mathbf{f_0})$ $|f(x,t)| \le C(1+|t|^{q(x)-1})$ for all $(x,t) \in \Omega \times \mathbb{R}$ with $C \ge 0$ and $1 < q(x) < p_M^*(x)$, where $p_M(x) = \max\{p_1(x), p_2(x)\}$, for all $x \in \overline{\Omega}$, and $p_M^*(x)$ is the critical exponent of p_M.
- $(\mathbf{f_1})$ $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{\frac{p_M^+}{1-\mu}}} = +\infty$, uniformly for a.e. $x \in \Omega$, where $\mu = \max\{\mu_1, \mu_2\}$ and $F(x,t) = \int_0^t f(x,s)\,ds$.
- $(\mathbf{f_2})$ There exists $\theta \ge 1$ such that $\theta G(x,t) \ge G(x,st)$ for $(x,t) \in \Omega \times \mathbb{R}$ and $s \in [0,1]$, where $G(x,t) = f(x,t)t - \frac{p_M^+}{1-\mu}F(x,t)$.
- $(\mathbf{f_3})$ $\lim_{t\to 0} \frac{F(x,t)}{|t|^{p_M^+}} = 0$, uniformly for a.e. $x \in \Omega$.

Now, we are ready to state our main result:

Theorem 1.1. *Suppose that the conditions* $(\mathbf{H_0})$-$(\mathbf{H_2})$ *and* $(\mathbf{f_0})$-$(\mathbf{f_3})$ *hold true, then problem* (1.1) *has at least one nontrivial weak solution.*

2. Notations and preliminaries

Let Ω be a bounded domain of $\mathbb{R}^N$, denote $C_+(\overline{\Omega}) = \{p(x);\ p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \forall x \in \overline{\Omega}\}$, $p^+ = \max\{p(x);\ x \in \overline{\Omega}\}$, $p^- = \min\{p(x);\ x \in \overline{\Omega}\}$;, $L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R}$ measurable and $\int_\Omega |u(x)|^{p(x)}\,dx < \infty\}$, with the *Luxemburg* norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\}$.

Proposition 2.1 (See [3])**.** *The space* $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ *is separable, uniformly convex, reflexive and its conjugate space is* $L^{q(x)}(\Omega)$ *where* $q(x)$ *is the conjugate function of* $p(x)$*, i.e.,* $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$*, for all* $x \in \Omega$*. For* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$*, we have* $\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)}$.

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as $W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) :\ D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \le k\}$, where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_N^{\alpha_N}}u$, with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space

$W^{k,p(x)}(\Omega)$ equipped with the norm $\|u\|_{k,p(x)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x)}$, also becomes a separable and reflexive Banach space. For more details, we refer the reader to [2, 3]. For any $k \geq 1$, denote

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if} \quad kp(x) < N, \\ +\infty & \text{if} \quad kp(x) \geq N \end{cases}$$

Proposition 2.2 (See [3]). *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$. If we replace $\leq$ with $<$, the embedding is compact.*

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Note that the weak solutions of problem (1.1) are considered in the generalized Sobolev space $X = X_1 \bigcap X_2$ equipped with the norm $\|u\|_r = \|u\|_{p_1} + \|u\|_{p_2}$, where $X_i = \left(W^{2,p_i(x)}(\Omega) \cap W_0^{1,p_i(x)}(\Omega)\right)$, $i = 1, 2$ and $\|u\|_r = \inf\{\mu > 0 : \int_\Omega \left|\frac{\Delta u(x)}{\mu}\right|^{r(x)} dx \leq 1\}$ equipped with the norm $\|u\| = \|u\|_{p_i}$.

Remark 2.3. According to [4], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $|\Delta \cdot|_{p(x)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $|\Delta \cdot|_{p(x)}$ are equivalent.

We consider the functional $\rho(u) = \int_\Omega |\Delta u|^{p(x)} dx$ and give the following fundamental proposition.

Proposition 2.4 (See [1]). *For $u \in X$ and $\{u_n\} \subset X$, we have*

(1) $\|u\| < 1(respectively = 1; > 1) \Longleftrightarrow \rho(u) < 1(respectively = 1; > 1)$;

(2) $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;

(3) $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;

(4) $\|u_n\| \to 0 \ (respectively \to \infty) \Longleftrightarrow \rho(u_n) \to 0 \ (respectively \to \infty)$.

Now, we recall the definition of the (C)-condition and then state a deformation lemma, which is fundamental in order to get some min-max theorems.

Definition 2.5. [7] let X be a Banach space and $J \in C^1(X, \mathbb{R})$. Given $c \in \mathbb{R}$, we say that J satisfies the Cerami c condition (we denote condition $(C)_c$), if

(**i**) any bounded sequence $\{u_n\} \subset X$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ has a convergent subsequence.

(**ii**) there exist constant $\delta, R, \beta > 0$ such that $\|J'(u)\|\|u\| \geq \beta$ for all $u \in J^{-1}([c-\delta, c+\delta])$ with $\|u\| \geq R$. If $J \in C^1(X, \mathbb{R})$ satisfies condition $(C)_c$ foe every $c \in \mathbb{R}$, we say that J satisfies condition (C).

Lemma 2.6. [8] *Let X be a Banach space, $J \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$, be such that $\|e\| > r$ and $b := \inf_{\|u\|=r} J(u) > J(0) \geq J(e)$. If J satisfies the condition (C_c) with $c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$, and $\Gamma := \gamma \in C([0,1], X) | \gamma(0) = 0, \gamma(1) = e$, then c is a critical value of J.*

3. Proof of Theorem 1.1

Define $I_{p_1(x)} := \int_\Omega \frac{1}{p_1(x)} |\Delta u|^{p_1(x)}\, dx$ and $I_{p_2(x)} := \int_\Omega \frac{1}{p_2(x)} |\Delta u|^{p_2(x)}\, dx$. The Euler-Lagrange functional associated to (1.1) is

$$J(u) = \widehat{M_1}(I_{p_1(x)}) + \widehat{M_2}(I_{p_2(x)}) - \int_\Omega F(x,u)\, dx.$$

Moreover, the derivative of J is given by

$$\begin{aligned}\langle J'(u), v\rangle = M_1(I_{p_1(x)}) &\int_\Omega |\Delta u|^{p_1(x)-2} \Delta u \Delta v\, dx \\ &+ M_2(I_{p_2(x)}) \int_\Omega |\Delta u|^{p_2(x)-2} \Delta u \Delta v\, dx - \int_\Omega f(x,u) v\, dx,\end{aligned}$$

for all $u, v \in X$. Then we know that the weak solution of (1.1) corresponds to the critical point of the functional J. These two Lemmas lead us to get the proof of our main result:

Lemma 3.1. *If* $(\mathbf{H_0})$- $(\mathbf{H_2})$ *and* $(\mathbf{f_0})$- $(\mathbf{f_2})$ *hold, then J satisfies the Cerami condition.*

Lemma 3.2. *If* $(\mathbf{H_0})$- $(\mathbf{H_1})$ *and* $(\mathbf{f_1})$, $(\mathbf{f_3})$ *hold true, then all the assertions in Lemma 2.6 are satisfied.*

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SOME FIXED POINT THEOREMS FOR Γ-WARDOWSKI-GERAGHTY CONTRACTIONS IN Γ-B-METRIC SPACES

BABAK MOHAMMADI*, ABDOLLAH DINMOHAMMADI, VAHID PARVANEH

Department of Mathematics, Marand Branch, Islamic Azad University, Marand, Iran
babakmohammadi28@yahoo.com
Department of Mathematics, Buein Zahra Thechnical University, Buein Zahra, Qazvin, Iran
dinmohammadi@bzte.ac.ir
Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran
zam.dalahoo@gmail.com

Abstract. Motivated by Wardowski [Fixed Point Theory Appl. 2012:94, 2012] we introduce and study a new contraction and a new generalized b-metric space called α-β-F-Γ contraction and Γ-b-metric space respectively to prove a fixed point result as a generalization of the Banach contraction principle. Moreover, we discuss some illustrative contractions to highlight the realized improvements.

1. Introduction

After Bakhtin [4] and Czerwik [5, 6] introduced b-metric spaces, many authors attempted to generalize this practical concept. Kamran and others [7] presented controlled b-metric spaces. Following that, double-controlled b-metric spaces were introduced. Generalized b-metric spaces were also introduced by Parvaneh and Ghoncheh [9]. They used a function instead of the coefficient s, which was always above the half-tone of the first and third

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quarters and did not coincide with its inverse except at zero. This is a major limitation. In this article, by removing this limitation, we present a new category of generalized b-metric spaces, which will certainly be of interest to many researchers in the field of fixed point theory.

The most well-known conclusion in fixed point theory, the Banach contractive principle or Banach fixed point theorem, states that every contractive mapping in a complete metric space has a unique fixed point. By applying various types of contractive conditions in different spaces, this result can be generalized in a huge number of ways. A fixed point conclusion was demonstrated recently as a generalization of the Banach contraction principle by Wardowski [1], who also established a new contraction known as the F-contraction.

One of the interesting results which also generalizes the Banach contraction principle was given by Samet *et al.* [2] by defining α-ψ-contractive and α-admissible mappings.

Definition 1.1. [2] Let T be a self-mapping on a set X and let $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 1.2. [3] Let $f : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that f is a triangular α-admissible mapping if

(T1) $\alpha(x, y) \geq 1 \quad \text{implies} \quad \alpha(fx, fy) \geq 1, \quad x, y \in X;$

(T2) $\begin{cases} \alpha(x, z) \geq 1 \\ \alpha(z, y) \geq 1 \end{cases} \quad \text{implies} \quad \alpha(x, y) \geq 1, \quad x, y, z \in X.$

Lemma 1.3. [3] *Let f be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Define sequence $\{x_n\}$ by $x_n = f^n x_0$. Then*

$$\alpha(x_m, x_n) \geq 1 \textit{ for all } m, n \in \mathbb{N} \textit{ with } m < n.$$

Definition 1.4. [5] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric if, for all $x, y, z \in X$,

(b_1) $d(x, y) = 0$ iff $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

Substituting the coefficient s by a function $\Omega : [0, \infty) \to [0, \infty)$ with some constraints, a two-variable function $\alpha : X \times X \to [0, \infty)$ and two two-variable functions $\alpha_1, \alpha_2 : X \times X \to [0, \infty)$ we have extended b-metric spaces, controlled b-metric spaces and double controlled b-metric spaces, respectively. In this paper, we introduce the concept of α-β-F-Γ-contractions and obtain some fixed point results in Γ-b-metric spaces. Our results extend those of Wardowski and several other authors.

2. Fixed point results for α-admissible β-FG-contractions

Removing some constraints on function Ω we have the following extension of the concept of b-metric space.

Definition 2.1. Let X be a (nonempty) set. A function $d : X \times X \to [0, \infty)$ is a Γ-b-metric if there exists a continuous increasing mapping $\Gamma : [0, \infty) \to [0, \infty)$ satisfying $\lim_{n\to\infty} \Gamma(t_n) = 0$ iff $\lim_{n\to\infty} t_n = 0$ such that for all $x, y, z \in X$,

(b_1) $d(x, y) = 0$ iff $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) $\Gamma[d(x, z)] \leq s \cdot \Gamma[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a Γ-b-metric space.

Note that a b-metric need not be a continuous function. So, we have this fact about Γ-b-metric spaces.

Example 2.2. Taking $\Gamma(x) = \operatorname{arcsinh} x$, the Γ-triangle inequality will be as follows:

$$\sinh^{-1}[d(x, z)] \leq s \cdot \sinh^{-1}[d(x, y) + d(y, z)].$$

So,

$$\begin{aligned}
[d(x, z)] &\leq \sinh[s \cdot \sinh^{-1}[d(x, y) + d(y, z)]] \\
&= \frac{e^{2s\cdot\ln[[d(x,y)+d(y,z)]+\sqrt{[d(x,y)+d(y,z)]^2+1}]} - 1}{2e^{s\cdot\ln[[d(x,y)+d(y,z)]+\sqrt{[d(x,y)+d(y,z)]^2+1}]}} \\
&= \frac{\left[[d(x, y) + d(y, z)] + \sqrt{[d(x, y) + d(y, z)]^2 + 1}\right]^{2s} - 1}{2\left[[d(x, y) + d(y, z)] + \sqrt{[d(x, y) + d(y, z)]^2 + 1}\right]^{s}}.
\end{aligned}$$

Example 2.3. Having any b-metric d_b on a nonemty set X, the function $d : X \times X \to [0, \infty)$ with $d(x, y) = \Upsilon(d_b(x, y))$, where $\Upsilon : [0, \infty) \to [0, \infty)$ is a invertible subadditive mapping, is a Γ-b-metric on X with $\Gamma(x) = \Upsilon^{-1}(x)$.

Example 2.4. Having any b-metric d_b on a nonemty set X, the function $d : X \times X \to [0, \infty)$ with $d(x, y) = sinh(d_b(x, y))$ is a Γ-b-metric on X with $\Gamma(x) = \sinh^{-1}(x)$.

Lemma 2.5. *Let (X, d) be a Γ-b-metric, and suppose that $\{x_n\}$ and $\{y_n\}$ are Γ-b-convergent to x and y, respectively. Then we have*

$$\frac{1}{s^2}\Gamma[d(x, y)] \leq \Gamma[\liminf_{n\to\infty} d(x_n, y_n)] \leq \Gamma[\limsup_{n\to\infty} d(x_n, y_n)] \leq s^2 \cdot \Gamma d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have,

$$\frac{1}{s}\Gamma[d(x,z)] \leq \liminf_{n\to\infty} \Gamma[d(x_n,z)] \leq \limsup_{n\to\infty} \Gamma[d(x_n,z)] \leq s \cdot \Gamma[d(x,z)].$$

Let $s > 1$ be a fixed real number. We will consider the following classes of functions.

Δ_F will denote the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that

(Δ_1) F is continuous and strictly increasing;
(Δ_2) for each sequence $\{t_n\} \subseteq R^+$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} F(t_n) = -\infty$.

$\Delta_{F,\beta}$ will denote the set of pairs (F, β), where $F : \mathbb{R}^+ \to \mathbb{R}$ and $\beta : [0,\infty) \to [0,1)$, such that

(Δ_3) for each sequence $\{t_n\} \subseteq R^+$, $\limsup_{n\to\infty} F(t_n) \geq 0$ if and only if $\limsup_{n\to\infty} t_n \geq 1$.
(Δ_4) for each sequence $\{t_n\} \subseteq [0,\infty)$, $\limsup_{n\to\infty} \beta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$;
(Δ_5) for each sequence $\{t_n\} \subseteq R^+$, $\sum_{n=1}^{\infty} F(\beta(t_n)) = -\infty$;

Definition 2.6. Let (X, d) be a Γ-b-metric space and let T be a self-mapping on X. Also suppose that $\alpha : X \times X \to [0,\infty)$ be a function. We say that T is an α-β-F-Γ-contraction if for all $x, y \in X$ with $1 \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$ we have

$$F\big(s\Gamma[d(Tx,Ty)]\big) \leq F\big(\Gamma[M_{s,\Gamma}(x,y)]\big) + F\big(\beta(M_{s,\Gamma}(x,y))\big), \tag{2.1}$$

where $F \in \Delta_F$, $(F, \beta) \in \Delta_{F,\beta}$ and

$$M_{s,\Gamma}(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{\Gamma^{-1}\big[\frac{d(x,Ty)+d(y,Tx)}{s}\big]}{2}\right\}. \tag{2.2}$$

Now we state and prove our main result of this section.

Theorem 2.7. *Let (X, d) be a complete Γ-b-metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *T is a triangular α-admissible mapping;*
(ii) *T is an α-β-F-Γ-contraction;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iv) *T is α-continuous.*

Then T has a fixed point. Moreover, T has a unique fixed point if $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$.

Taking $F(t) = \ln t$, $\beta(x) = k$ where $k \in (0, 1)$, and putting $\Gamma(x) = x^2$ in the above theorem, we obtain a generalization of the Banach contraction principle in the setup of Γ-b-metric spaces.

Corollary 2.8. *Let (X, d) be a complete Γ-b-metric space. Let T be a self-mapping on X and $\alpha : X \times X \to [0, \infty)$ be a function such that the mapping T satisfies the following assertions:*

(i) *T is a triangular α-admissible mapping;*

(ii)
$$d(Tx, Ty) < \sqrt{k/s}\big(M_{s,\Gamma}(x, y)\big),$$
for all $x, y \in X$ with $1 \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$;

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iv) *T is α-continuous.*

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$.

In the following theorem we replace the α-continuity of T by another condition (iv').

Theorem 2.9. *Let (X, d) be a complete Γ-b-metric space. Let $T : X \to X$ be a self-mapping and suppose that $\alpha : X \times X \to [0, \infty)$ be a function such that:*

(i) *T is a rectangular α-admissible mapping;*

(ii) *T is an α-β-F-Γ-contraction;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iv') *if $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ with $x_n \to x$ as $n \to \infty$, then $1 \leq \alpha(Tx_n, Tx)$ holds for all $n \in \mathbb{N}$.*

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$.

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A NEW EXTENSION OF G-METRIC SPACES

BABAK MOHAMMADI*, ABDOLLAH DINMOHAMMADI, VAHID PARVANEH

Department of Mathematics, Marand Branch, Islamic Azad University, Marand, Iran
babakmohammadi28@yahoo.com
Department of Mathematics, Buein Zahra Thechnical University, Buein Zahra, Qazvin, Iran
dinmohammadi@bzte.ac.ir
Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran
zam.dalahoo@gmail.com

Abstract. In this paper we present some fixed point results for Banach and Kannan contractive mappings in the setup of sequential G-metric spaces. This new structure, is a generalization of both G-metric spaces and G_b-metric spaces.

1. Introduction

Over the past few decades, experts in fixed point theory have generalized the traditional metric structure multiple times. In order to prove fixed point theorems utilizing various contractive, expansive, or non-expansive type mappings, various topologically organized spaces are necessary. Fixed point theory is becoming more and more popular in the mathematical community, particularly among young academics working on functional analysis, as a result of the existence of such fascinating spaces and the different types of applications of fixed point theorems therein. All existing generalizations of the concept of metric space emphasize the change in several components: one is the change of the domain of the metric mapping from the Cartesian

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product of a set in itself to the Cartesian product of higher orders; the other is the change of triangle inequality; the other is the elimination of the symmetry property of the meter, which leads to the definition of quasi-metric spaces; and another is the change in the first condition of the meter, which expresses the zero distance of an element to itself, or a change in several cases at the same time. However, these changes, either individually or in combination, lead to the production of new generalized metric spaces with different topological behaviors. However, the main task is to obtain previously proven fixed-point results under weaker conditions and with fewer assumptions than before. It goes without saying that the results obtained in this field are impressive and thought-provoking.

Here, we define a few generic spaces that are important to our research.

Definition 1.1. (b-metric space)[4, 5] Let Λ be a nonempty set and s be a real number satisfying $s \geq 1$. A function $\rho_b : \Lambda \times \Lambda \to \mathbb{R}^+$ is a $b-$metric on Λ if:
1. $\rho_b(\iota, \kappa) = 0$ if and only if $\iota = \kappa$;
2. $\rho_b(\iota, \kappa) = \rho_b(\kappa, \iota)$ for all $\iota, \kappa \in \Lambda$;
3. $\rho_b(\iota, z) \leq s[\rho_b(\iota, \kappa) + \rho_b(\kappa, z)]$ for all $\iota, \kappa, z \in \Lambda$.

The space (Λ, ρ_b) is called a $b-$metric space.

Let Λ be a non-empty set and $\rho_g : \Lambda \times \Lambda \to [0, \infty]$ be a mapping. For any $\iota \in \Lambda$, let us define the set

$$C(\rho_g, \Lambda, \iota) = \{\{\iota_n\} \subset \Lambda : \lim_{n\to\infty} \rho_g(\iota_n, \iota) = 0\}. \tag{1.1}$$

Definition 1.2. (JS-metric space)[6] Let $\rho_g : \Lambda \times \Lambda \to [0, \infty]$ be a mapping which satisfies:
1. $\rho_g(\iota, \kappa) = 0$ implies $\iota = \kappa$;
2. for every $\iota, \kappa \in \Lambda$, we have $\rho_g(\iota, \kappa) = \rho_g(\kappa, \iota)$;
3. if $(\iota, \kappa) \in \Lambda\times\Lambda$ and $\{\iota_n\} \in C(\rho_g, \Lambda, \iota)$ then $\rho_g(\iota, \kappa) \leq p \limsup_{n\to\infty} \rho_g(\iota_n, \kappa)$,

for some $p > 0$.

The pair (Λ, ρ_g) is called a generalized metric space, usually known as $JS-$metric space or sequential metric space.

The concept of generalized metric space, or a G-metric space, was introduced by Mustafa and Sims [3].

Definition 1.3. (G-Metric Space, [3]) Let X be a nonempty set and G: $X \times X \times X \to R^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$;

(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables);

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Aghajani et al. in [2] introduce the concept of generalized b-metric spaces, or $\mathrm{G_b}$-metric spaces as follows

Definition 1.4. (G_b-Metric Space) Let X be a nonempty set and G: $X \times X \times X \to R^+$ be a function satisfying the following properties:
(G1) $G(x, y, z) = 0$ if $x = y = z$;
(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables);
(G5) $G(x, y, z) \leq s(G(x, a, a) + G(a, y, z))$, for all $x, y, z, a \in X$ and for some $s \geq 1$. (rectangle inequality).
Then, the function G is called a G_b-metric on X and the pair (X, G) is called a G_b-metric space.

Then they present some basic properties of $\mathrm{G_b}$-metric spaces. In this paper, we introduce sequential G-metric and obtain some fixed point theorems for Banach and Kannan contractive mappings in sequential G-metric spaces. This results generalize and modify several comparable results in the literature.

2. Main results

The following is the definition of sequential G-metric spaces.

Definition 2.1. Let X be a nonempty set and $p > 0$ be a given real number. Suppose that a mapping G: $X \times X \times X \to \mathbb{R}^+$ satisfies:
(G_b1) $G(x, y, z) = 0$ implies $x = y = z$,
(G_b2) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G_b3) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),
(G_b4) $G(x, y, z) \leq p \cdot \limsup\limits_{n \longrightarrow \infty} G(x_n, y, z)$ for all $x, y, z \in X$ and for all $\{x_n\}$ which is G-convergent to x.

Then G is called a sequential G-metric and the pair (X, G) is called a sequential G-metric space.

Inspired from [1] we have the following example.

Example 2.2. Let $\Lambda = \mathbb{N}$ and the metric $\sigma : \Lambda^2 \to [0, \infty)$ be defined by

$$\begin{cases} \sigma(1,1) = 0; & \\ \sigma(n,n) = e - 1, & \text{for } n \geq 2; \\ \sigma(1,n) = \sigma(n,1) = e^{\frac{1}{n+1}} - 1, & \text{for } n \geq 2; \\ \sigma(n,m) = \sigma(m,n) = e^{mn} - 1, & \text{for all } n, m \geq 2 \text{ with } n \neq m. \end{cases}$$

Then, σ is a sequential metric on Λ.
Define G: $X \times X \times X \to \mathbb{R}^+$ by $G(x, y, z) = \max\{\sigma(x, y), \sigma(y, z), \sigma(z, x)\}$.

Remark 2.3. Any G-metric space, G_b-metric space are also sequential G-metric spaces.

Proposition 2.4. *In a sequential G-metric space (Λ, G) if a sequence $\{a_n\}$ is convergent, then it converges to a unique element in Λ.*

Proof. Suppose $a, b \in X$ be such that $a_n \to a$ and $a_n \to b$ as $n \to \infty$. Then we have, $G(a, b, b) \leq p\left(\limsup_{n\to\infty} G(a_n, b, b)\right)$ implying that $G(a, b, b) \leq 0$ *i.e.*, $a = b$. □

Proposition 2.5. *Let (X, G) be a sequential G-metric space and $\{a_n\} \subset X$ converges to some $a \in X$. Then $G(a, a, a) = 0$.*

Proof. Since $\{a_n\}$ converges to $a \in X$, so $\lim_{n\to\infty} G(a_n, a, a) = 0$. Therefore we have $G(a, a, a) \leq p(\limsup_{n\to\infty} G(a_n, a, a)) = 0$ which implies $G(a, a, a) = 0$. □

Proposition 2.6. *Let $\{a_n\}$ be a Cauchy sequence in a sequential G-metric space (X, G). If $\{a_n\}$ has a convergent sub-sequence $\{a_{n_k}\}$ which converges to $a \in X$, then $\{a_n\}$ also converges to $a \in X$.*

Proof. From condition (G_b5) of Definition 2.1 we have

$$G(a_n, a, a_n) \leq p\left(\limsup_{k\to\infty} G(a_n, a_{n_k}, a_n)\right)$$

which implies that $p^{-1}\left(G(a_n, a, a_n)\right) \leq \limsup_{k\to\infty} G(a_n, a_{n_k}, a_n)$ for all $n \in \mathbb{N}$.

Due to the Cauchyness of $\{a_n\}$ it follows that $\lim_{n,k\to\infty} G(a_n, a_{n_k}, a_n) = 0$ which implies that $G(a_n, a, a_n) \to 0$ as $n \to \infty$. Hence, $\{a_n\}$ converges to $a \in X$. □

Proposition 2.7. *In a sequential G-metric space (X, G), if a self mapping T is continuous at $a \in X$, then $\{Ta_n\} \to Ta$.*

Proof. Let $\epsilon > 0$ be given. Since T is continuous at a, there exists $\delta_\epsilon > 0$ such that $G(c, a, a) < \delta_\epsilon$ implies $G(Tc, Ta, Ta) < \epsilon$.

As $\{a_n\}$ converges to a, so for $\delta_\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(a_n, a, a) < \delta_\epsilon$ for all $n \geq N$. Therefore, for any $n \geq N$, $G(Ta_n, Ta, Ta) < \epsilon$ and thus $Ta_n \to Ta$ as $n \to \infty$. □

Remark 2.8.
1. In a metric space, a convergent sequence is always Cauchy, but it is not true in a sequential G-metric space. In Example 2.2, the sequence $\{n\}_{n\geq 2}$ converges to 1, but $G(o, n, m) = \max\{e^{on} - 1, e^{nm} - 1, e^{mo} - 1\} \not\to 0$ whenever $o, n, m \to \infty$.
2. In a metric space, if $\{a_n\}$ and $\{b_n\}$ are two sequences converging to a and b respectively then $G(a_n, b_n) \to G(a, b)$ as $n \to \infty$. But this does not always hold in a sequential G-metric space. In Example 2.2, let us consider two sequences $\{2n\}_{n\geq 1}$ and $\{2n+1\}_{n\geq 1}$ in G. Then both of these two sequences converge to $1 \in \Lambda$, but $G(2n, 2n+1, 2n+1) \not\to 0$ as $n \to \infty$.

3. A G-metric and G_b-metric is always a sequential G-metric space, but the converse is not true in general. For example, the metric G defined in Example 2.2 is not a G_b-metric (and so is not a G-metric). Since if it be a G_b-metric, then $G(n,m,1) \leq b(G(n,1,1)+G(1,m,1))$ for all $n,m \geq 2$ with $n \neq m$, then $\max\{e^{nm}-1, e^{\frac{1}{n+1}}-1, e^{\frac{1}{m+1}}-1\} \leq b(e^{\frac{1}{n+1}}-1+e^{\frac{1}{m+1}}-1)$ for all $n,m \geq 2$. Taking limit as $n \to \infty$ we get $\infty \leq 0$, which is a contradiction.

Definition 2.9. Let X be a sequential G-metric space. We define

$$d_G(x,y) = G(x,y,y) + G(x,x,y),$$

for all $x,y \in X$. It is easy to see that d_G defines a sequential metric on X, which we call it the sequential metric associated with G.

Theorem 2.10. *Let (X,G) be a complete sequential G-metric space and $\Upsilon : G \to G$ be a mapping so that:*
(i) $G(\Upsilon a, \Upsilon b, \Upsilon c) \leq \alpha G(a,b,c)$ for all $a,b,c \in G$ and for some $\alpha \in (0,1)$,
(ii) there exists $a_0 \in G$ such that

$$\delta(G,\Upsilon,a_0) := \sup\left\{G\left(\Upsilon^i a_0, \Upsilon^j a_0, \Upsilon^k a_0\right) : i,j,k = 1,2,\cdots\right\} < \infty.$$

Then Υ has at least one fixed point in G. Moreover, if a and b are two fixed points of Υ in G with $G(a,b,b) < \infty$, then $a = b$.

Theorem 2.11. *Let (X,G) be a complete sequential G-metric space and $\Upsilon : G \to G$ such that:*
(i) $G(\Upsilon a, \Upsilon b, \Upsilon c) \leq \gamma[G(a,\Upsilon a,\Upsilon a) + G(b,\Upsilon b,\Upsilon b) + G(c,\Upsilon c,\Upsilon c)]$ for all $a,b,c \in G$ and for some $\gamma \in (0,\frac{1}{3})$,
(ii) there exists $a_0 \in G$ such that

$$\delta(G,\Upsilon,a_0) := \sup\left\{G\left(\Upsilon^i a_0, \Upsilon^j a_0,, \Upsilon^k a_0\right) : i,j,k = 1,2,\cdots\right\} < \infty.$$

Then the Picard iterating sequence $\{a_n\}$, $a_n = \Upsilon^n a_0$ for all $n \in \mathbb{N}$, converges to some $a \in G$. If $G(a,\Upsilon a,\Upsilon a) < \infty$, then a is a fixed point of Υ. Moreover if b is a fixed point of Υ in G such that $G(a,b,b) < \infty$ and $G(b,b,b) < \infty$ then $a = b$.

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Oral Presentation **$*$: Speaker**

ON THE POSITIVE OPERATORS AND LINEAR MAPS

LEILA NASIRI*AND MAHMOOD SHAKOORI

Department of Mathematics and computer science, Faculty of science, Lorestan University, Khorramabad, Iran.
leilanasiri468@gmail.com and mahmoodshakoori@gmail.com

Abstract. In this work, we first prove an numerical inequality. Then, we present some inequalities for positive operators and linear maps.

1. Introduction

In what follows, we denote by $\mathbb{B}(\mathbf{H}))$ the space of all bounded linear operators on a Hilbert space $(H, <.,.>)$. We say the operator A is positive if $< Ax, x >\geq 0$ for all $x \in \mathbf{H}$ and write $A \geq 0$, is invertible positive if $< Ax, x >> 0$ for all $x \in \mathbf{H}$ and write $A > 0$. For two selfadjoint operators $A, B \in \mathbb{B}(\mathbf{H})$, we say $A \geq B(A > B)$ if $A - B \geq 0$ $(A - B > 0)$, respectively. The adjoint of the operator A define by A^* and its absolute value by $|A|$, that is, $|A| = (A^*A)^{\frac{1}{2}}$. A linear map Φ is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$. For $A, B \in \mathbb{B}(\mathbf{H})$ such that A and B are invertible positive and $0 \leq \nu \leq 1$, we utilize the following notations to define the geometric mean $A\sharp_\nu B$ and the arithmetic mean $A\nabla_\nu B$, respectively,

$$A\#_\nu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}} \quad \textit{and} \quad A \bigtriangledown_\nu B = \nu A + (1-\nu)B.$$

For $A, B \in \mathbb{B}(\mathbf{H})$ such that are invertible positive and $0 \leq \nu \leq 1$, we have operator Young inequality as follows:

$$A\#_\nu B \leq A\nabla_\nu B. \tag{1.1}$$

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* Speaker.

The Lowner-Heinz theorem [6] states that if $A, B \in \mathbb{B}(\mathbf{H})$ are positive, then for $0 \leq p \leq 1$,

$$A \leq B \qquad \text{implies} \quad A^p \leq B^p. \tag{1.2}$$

In general (1.2) is not true for $p > 1$. M. Lin [5] reversed (1.1) using the Katorovich constant as follows: If $0 < m \leq A, B \leq M$ and Φ are a unital positive linear map. Then

$$\varphi^2(A\nabla B) \leq K^2(h)\varphi^2(A\#B), \varphi^2(A\nabla B) \leq K^2(h)(\varphi(A)\#\varphi(B))^2, \tag{1.3}$$

where $K(h) = \frac{(1+h)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.

2. Main results

Furuichi et al. in [4] showed that for $0 < x \leq 1$ and $0 \leq \nu \leq 1$, the following inequality holds:

$$m_\nu(x)x^\nu \leq (1-\nu) + \nu x, \tag{2.1}$$

where $m_\nu(x) = 1 + \frac{2^\nu \nu(1-\nu)(x-1)^2}{(x+1)^{1+\nu}}$ and $1 \leq m_\nu(x)$. The next Lemma is an refinement of (2.1).

Lemma 2.1. *Let $0 < x \leq 1$. If $0 \leq \nu \leq \frac{1}{2}$, then*

$$m_{2\nu}(\sqrt{x})x^\nu + \nu(\sqrt{x}-1)^2 \leq (1-\nu) + \nu x, \tag{2.2}$$

where $m_{2\nu}(\sqrt{x})$ defined as (2.1).

Proof. Letting $0 \leq \nu \leq \frac{1}{2}$. By an simple computation

$$(1-\nu) + \nu x - \nu(\sqrt{x}-1)^2 = 2\nu\sqrt{x} + (1-2\nu).$$

By applying (2.1) for the relation above,

$$m_{2\nu}(\sqrt{x})x^\nu \leq (1-2\nu) + 2\nu\sqrt{x}.$$

Therefore, (2.2) is proved. □

For a operator version of the inequality (2.2), see Theorem 2.2.

Theorem 2.2. *Let $A, B \in \mathbb{B}(H)$ are two invertible positive operators such that $0 < m \leq A \leq m' < M' \leq B \leq M$ or $0 < m \leq B \leq m' < M' \leq A \leq M$ for some positive real numbers m, m', M, M'. Then for $0 \leq \nu \leq \frac{1}{2}$*

$$A\nabla_\nu B \geq m_{2\nu}(\sqrt{h})A\sharp_\nu B + \nu(A\nabla B - A\sharp B), \tag{2.3}$$

where $m_{2\nu}(\sqrt{x})$ defined as (2.1).

Proof. The condition $0 < m \leq A \leq m' < M' \leq B \leq M$ ensure us that $Sp\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) \subset [h', h]$, where $h = \frac{M}{m}$ and $h = \frac{M'}{m'}$. On the other hand, (2.2) implies that

$$\min_{\sqrt{h'} \leq \sqrt{x} \leq \sqrt{h} \leq 1} m_{2\nu}(\sqrt{x})x^\nu + \nu(\sqrt{x}-1)^2 \leq (1-\nu) + \nu x, \tag{2.4}$$

If we set $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in the inequality (2.4) and use from the decreasing property $m_{2\nu}(\sqrt{x})$, the following inequality deduces

$$\begin{aligned}&\nu A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+(1-\nu)I\\ &\geq m_{2\nu}(\sqrt{h})\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu}+\nu(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I-2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}).\end{aligned} \tag{2.5}$$

Multiplying both sides of the inequality (2.5) by $A^{\frac{1}{2}}$, we can get to the desired result. Similarly, the condition $0 < m \leq B \leq m^{'} < M^{'} \leq A \leq M$ concludes the desired result. □

Remark 2.3. As $A\nabla B - A\sharp B \geq 0$. By (2.3), we have $A\nabla_{\nu}B \geq m_{2\nu}(\sqrt{h})A\sharp_{\nu}B$. From $m_{2\nu}(\sqrt{h}) \geq 1$, (**??**) is a refinement of (1.1).

2.1. The higher powers using positive mps.

Lemma 2.4. [3] *Let $A \in \mathbb{B}(H)$ be positive and Φ be a positive unital linear map. Then*

$$\Phi(A)^{-1} \leq \Phi\left(A^{-1}\right). \tag{2.6}$$

Lemma 2.5. [2]-[1] *Let $A, B \geq 0$. Then for $1 \leq r < \infty$*

$$\|AB\| \leq \frac{1}{4}\|A+B\|^{2}. \tag{2.7}$$

$$\|A^{r}+B^{r}\| \leq \|(A+B)^{r}\|. \tag{2.8}$$

Theorem 2.6. *Let $A, B \in \mathbb{B}(H)$ are two invertible positive operators such that $0 < m \leq A \leq m^{'} < M^{'} \leq B \leq M$ or $0 < m \leq B \leq m^{'} < M^{'} \leq A \leq M$ for some positive real numbers $m, m^{'}, M, M^{'}$ and Φ be a normalized positive linear map. Then for every $0 \leq \nu \leq \frac{1}{2}$*

$$\Phi^{2}(A\nabla_{\nu}B+\nu Mm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right)) \leq \left(\frac{K(h)}{m_{2\nu}(\sqrt{h})}\right)^{2}\Phi^{2}(A\sharp_{\nu}B), \tag{2.9}$$

$$\Phi^{2}(A\nabla_{\nu}B+\nu Mm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right)) \leq \left(\frac{K(h)}{m_{2\nu}(\sqrt{h})}\right)^{2}(\Phi(A)\sharp_{\nu}\Phi(B))^{p}, \tag{2.10}$$

where $K(h)=\frac{(h+1)^{2}}{4h}$ with $h=\frac{M}{m}$ is the Kantorovich constant and $m_{2\nu}(\sqrt{h})$ is as defined in (2.1).

Proof. It is trivial that$A+MmA^{-1} \leq M+m$and$B+MmB^{-1} \leq M+m$. An simple computation show that

$$\Phi\left(A\nabla_{v}B\right)+Mm\Phi\Big(A^{-1}\nabla_{v}B^{-1}\Big) \leq M+m. \tag{2.11}$$

By applying (2.7), (2.6), (2.3)and(2.11), respectively, one can check that

$$
\begin{aligned}
&\left\|\Phi(A\nabla_\nu B+\nu Mm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right))Mmm_{2\nu}(\sqrt{h})\Phi^{-1}(A\sharp_\nu B)\right\|\\
&\leq\frac{1}{4}\left\|\Phi(A\nabla_\nu B+\nu Mm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right))+Mmm_{2\nu}(\sqrt{h})\Phi^{-1}(A\sharp_\nu B)\right\|^2\\
&\leq\frac{1}{4}\left\|\Phi(A\nabla_\nu B+\nu Mm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right))+Mmm_{2\nu}(\sqrt{h})\Phi(A^{-1}\sharp_\nu B^{-1})\right\|^2\\
&=\frac{1}{4}\left\|\Phi(A\nabla_\nu B)+Mm\Phi\left(\nu\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right)+m_{2\nu}(\sqrt{h})(A^{-1}\sharp_\nu B^{-1})\right)\right\|^2\\
&\leq\frac{1}{4}\left\|\Phi(A\nabla_\nu B)+Mm\Phi(A^{-1}\nabla_\nu B^{-1})\right\|^2\\
&\leq\frac{(M+m)^2}{4}.
\end{aligned}
$$

This proves the inequality (2.9). The inequality (2.10) can prove similarly. □

Corollary 2.7. *Let $A,B\in\mathbb{B}(H)$ are two invertible positive operators such that $0<m\leq A\leq m'<M'\leq B\leq M$ or $0<m\leq B\leq m'<M'\leq A\leq M$ for some positive real numbers m,m',M,M' and Φ be a normalized positive linear map. Then for $p>0$ and every $0\leq\nu\leq\frac{1}{2}$*

$$\Phi^p(A\nabla_\nu B+\nu Mm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right))\leq\left(\frac{K(h)}{m_{2\nu}(\sqrt{h})}\right)^p\Phi^p(A\sharp_\nu B),\tag{2.12}$$

$$\Phi^p(A\nabla_\nu B+\nu Mm\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right))\leq\left(\frac{K(h)}{m_{2\nu}(\sqrt{h})}\right)^p(\Phi(A)\sharp_\nu\Phi(B))^p,\tag{2.13}$$

where $K(h)=\frac{(h+1)^2}{4h}$ with $h=\frac{M}{m}$ is the Kantorovich constant and $m_{2\nu}(\sqrt{h})$ is as defined in (2.1).

Proof. If $0<p\leq 2$, then $0<\frac{p}{2}\leq 1$. Thus, by Theorem 2.6, we obtain the desired results. Letting $p>2$. By (2.8) and the same method as used in Theorem 2.6 the inequalities above conclude. □

Remark 2.8. It is clear that $A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\geq 0$. Thus,

$$\Phi^p(A\nabla_\nu B)+(\nu Mm)^p\Phi^p(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\geq\Phi^p(A\nabla_\nu B).$$

In result,

$$
\begin{aligned}
\|\Phi^p(A\nabla_\nu B)\|&\leq\|\Phi^p(A\nabla_\nu B)+(\nu Mm)^p\Phi^p(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\|\\
&\leq\|(\Phi(A\nabla_\nu B)+(\nu Mm)\Phi(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))^p\|(by(2.8)).
\end{aligned}
$$

On the other hand, by (2.1)., $m_{2\nu}(\sqrt{h})\geq 1$. This shows that hand-left side of (2.12) and (2.13) is a norm refinement of (1.3) and hand-right side of (2.12) and (2.13) are tighter than (1.3).

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MULTIPLE SOLUTIONS FOR AN ANISOTROPIC VARIABLE EXPONENT PROBLEM WITH NEUMANN BOUNDARY CONDITION

[1]T. NOROUZI GHARA*, [1]G.A. AFROUZI AND [2]S. KHADEMLOO

[1]*Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran*
[2]*Department of Basic Sciences, Babol (Noushirvani) University of Technology Babol, Iran*
tahere.noruzi5@gmail.com, afrouzi@umz.ac.ir, s.khademloo@nit.ac.ir

Abstract. By using variational methods and critical point theory, we establish the existence of multiple solutions for a Neumann problem. We prove the existence by applying the theory of variable exponent Sobolev spaces.

1. Introduction

In the present paper, we want to establish the existence of multiple solutions for the following problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + h(x) \sum_{i=1}^{N} a_i(x, u) = \lambda f(x, u) & in\ \Omega, \\ \sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\nu_i = 0 & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N(\ N \geq 3)$ with smooth boundary $\partial\Omega$, ν_i of the outer normal unit vector to $\partial\Omega$, λ is a positive parameter, while

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$f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and $h(x)$ is a positive function such that $h(.) \in L^{\infty}(\Omega)$ and

$$h^{-} = \operatorname{ess}\inf_{x\in\Omega} h(x) > 0, \tag{1.2}$$

and

$$h^{+} = \operatorname{ess}\sup_{x\in\Omega} h(x) > 0. \tag{1.3}$$

2. PRELIMINARIES

In this section we recall some definition and the main properties of the spaces with variable exponents together with some results that we need for the proof of our main results.

Define

$$C_{+}(\Omega) := \{p \;:\; p \in C(\overline{\Omega}) \text{ and } p(x) > 1,\ \forall x \in \overline{\Omega}\}.$$

For $p \in C_{+}(\overline{\Omega})$, we introduce the Lebesgue space with variable exponent defined by

$$L^{p(x)}(\Omega) = \{u \;:\; u \in S(\Omega),\ \int_{\Omega} |u(x)|^{p(x)} dx < \infty\},$$

where $S(\Omega)$ denotes the set of all measurable real functions on Ω.

This space, endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\tau > 0 \;:\; \int_{\Omega} |\frac{u(x)}{\tau}|^{p(x)} dx \leqslant 1\},$$

is a separable and reflexive Banach space. We refer to [4, **?**, 7, 8] for the elementary properties of these spaces.

Let

$$p^{+} = \max_{x\in\overline{\Omega}} p(x), \qquad p^{-} = \min_{x\in\overline{\Omega}} p(x).$$

To recall the definition of the isotropic Sobolev space with variable exponent, $W^{1,p(x)}$, we set

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \;:\; |\nabla u| \in L^{p(x)}(\Omega)\},$$

endowed with the norm

$$\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}. \tag{2.1}$$

The space $W^{1,p(x)}(\Omega)$, equipped with the norm 2.1 becomes a separable, reflexive and uniformly convex Banach space. See for more details [1].

For $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_h = \inf\{\eta > 0 \;:\; \int_{\Omega} (|\frac{\nabla u}{\eta}|^{p(x)} + h(x)|\frac{u}{\eta}|^{p(x)}) dx \leqslant 1\}. \tag{2.2}$$

We assume in the sequel that Ω is a bounded open domain in $\mathbb{R}^N$ and we denote by

$$\overrightarrow{p}(.) : \overline{\Omega} \to \mathbb{R}^N$$

the vectorial function

$$\overrightarrow{p}(.) = (p_1(.), ..., p_N(.)).$$

We define $W^{1,\overrightarrow{p}(.)}(\Omega)$, the anisotropic variable exponent Sobolev space with respect to the norm

$$\|u\|_{\overrightarrow{p}(.)} = \|u\|_{W^{1,\overrightarrow{p}(.)}(\Omega)} = \sum_{i=1}^{N} \inf\{\sigma > 0\,;\, (\int_\Omega |\frac{\partial_{x_i} u}{\sigma}|^{p_i(x)}dx + \int_\Omega h(x)|\frac{u}{\sigma}|^{p_i(x)}dx) \leqslant 1\}. \tag{2.3}$$

It was argued in [5] that $W^{1,\overrightarrow{p}(.)}(\Omega)$ is a reflexive Banach space and a seperable space.

In [2] Bonnano proposed the following innovative theorems for the study of nonlinear problems:

Theorem 2.1. *([2] Theorem 5.2) Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Let $I_\lambda = \Phi - \lambda\Psi$ and for fix $r > \inf_X \Phi$ let φ be the function defined as*

$$\varphi(r) := \inf_{v\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) - \Psi(u)}{r - \Phi(v)}.$$

Then, for each $\lambda \in]0, \frac{1}{\varphi(r)}[$ there is $u_{0,\lambda} \in \Phi^{-1}(]-\infty, r[)$ such that $I_\lambda(u_{(0,\lambda)}) \leqslant I_\lambda(u)$ for all $u \in \Phi^{-1}(]-\infty, r[)$ and $I'_\lambda(u_{(0,\lambda)}) = 0$.

Now, we state our first main results as follows.

Theorem 2.2. *Assume that*

$$\sup_{\gamma>0} \frac{\min\{k_1, ..., k_n\}\gamma^{p_-^-}}{\int_\Omega \sup_{|t|\leqslant\gamma} F(x,t)dx} > p_+^+ c^{p_-^-}, \tag{2.4}$$

*where c is the constant defined in (**??**). Then the problem (1.1) admits at least one weak solution in $W^{1,\overrightarrow{p}(.)}(\Omega)$.*

Now, we state second main results to find three weak solutions for the problem (1.1). Our approach is the following problem:

Theorem 2.3 ([2, Theorem 7.1])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ bounded from below. Assume that there is $r \in]inf_X\Phi, sup_X\Psi[$ such that*

$$\varphi(r) < \rho(r),$$

where

$$\varphi(r) := \inf_{v\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) - \Psi(v)}{r - \Phi(v)},$$

and

$$\rho(r) := \sup_{v\in\Phi^{-1}(]r,\infty[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{\Phi(v) - r}.$$

and for each $\lambda \in \left]\frac{1}{\rho(r)}, \frac{1}{\varphi(r)}\right[$ *the function* $I_\lambda = \Phi - \lambda\Psi$ *is bounded from below and satisfies (PS)-condition.*
Then, for each $\lambda \in \left]\frac{1}{\rho(r)}, \frac{1}{\varphi(r)}\right[$ *the function* I_λ *admits at least three critical points.*

Theorem 2.4. *Assume that c be a positive constants with*

$$\frac{\int_\Omega \sup_{|t|\le c} F(x,t)dx}{r} < \frac{\int_\Omega F(x,\delta)dx}{\zeta c^{p_+^+}}, \tag{2.5}$$

and

$$0 < r < h^- \min\{k_1,\dots,k_N\} \frac{\delta^{p_-^-}}{p_+^+} meas(\Omega) \tag{2.6}$$

Then, for each parameter λ *belonging to*

$$\Lambda_{(r,\delta)} := \tag{2.7}$$

$$\left]\frac{\zeta c^{p_+^+}}{\int_\Omega F(x,\delta)dx}, \frac{r}{\int_\Omega \sup_{|t|\le c} F(x,t)dt}\right[,$$

the problem (1.1) *possesses at least three distinct weak solutions in* $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

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Oral Presentation **$*$: Speaker**

NON-LINEAR MAPS PRESERVING THE PSEUDO SPECTRUM OF OPERATOR PRODUCTS

ROHOLLAH PARVINIANZADEH* AND MEYSAM ASADIPOUR

Department of Pure Mathematics, University of Yasouj, Yasouj 75918, Iran
r.parvinian@yu.ac.ir; Asadipour@yu.ac.ir

Abstract. Let $B(H)$ be the algebra of all bounded linear operators on infinite-dimensional complex Hilbert space H. Fix $\epsilon > 0$ and $T \in B(H)$, let $\sigma_\epsilon(T)$ denote the ϵ-pseudo spectrum of T. In this note, we show if the surjective map φ on $B(H)$ satisfies

$$\sigma_\epsilon(TS - ST^*) = \sigma_\epsilon(\varphi(T)\varphi(S) - \varphi(S)\varphi(T)^*),\ (T, S \in B(H)),$$

then there exists a unitary operator $U \in B(H)$ such that $\varphi(T) = \mu UTU^*$ for every $T \in B(H)$, where $\mu \in \{-1, 1\}$.

1. Introduction

Throughout this paper, $B(H)$ stands for the algebra of all bounded linear operators acting on an infinite dimensional complex Hilbert space $(H, \langle,\rangle)$ and its unit will be denoted by I. Let $B_s(H)$ (resp. $B_a(H)$) be the real linear space of all self-adjoint (resp. anti-self-adjoint) operators in $B(H)$. For an operator $T \in B(H)$, the adjoint and the spectrum of T are denoted by T^* and $\sigma(T)$, respectively. For $\epsilon > 0$, the ϵ-pseudo spectrum of T, $\sigma_\epsilon(T)$, is defined by $\sigma_\epsilon(T) = \cup\{\sigma(T + A) : A \in B(H), \|A\| \leq \epsilon\}$ and coincides with the set

$$\{\lambda \in \mathbb{C} : \|(\lambda I - T)^{-1}\| \geq \epsilon^{-1}$$

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$*$ Speaker.

with the convention that $\|(\lambda I - T)^{-1}\| = \infty$ if $\lambda \in \sigma(T)$. It is a compact subset of $\mathbb{C}$ and contains $\sigma(T)$, the spectrum of T. Unlike the spectrum, which is a purely algebraic concept, the ϵ-pseudo spectrum depends on the norm. The ϵ-pseudo spectral radius of T, $r_\epsilon(T)$, is given by

$$r_\epsilon(T) = sup\{|\lambda| : \lambda \in \sigma_\epsilon(T)\}.$$

Pseudo spectra are a useful tool for analyzing operators, furnishing a lot of information about the algebraic and geometric properties of operators and matrices. They play a very natural role in numerical computations, especially in those involving spectral perturbations. The book [5] gives an extensive account of the pseudo spectra, as well as investigations and applications in numerous fields.

Linear preserver problems, in the most general setting, demands the characterization of maps between algebras that leave a certain property, a particular relation, or even a subset invariant. In all cases that have been studied by now, the maps are either supposed to be linear, or proved to be so. This subject is very old and goes back well over a century to the so-called first linear preserver problem, due to Frobenius [4], who characterized linear maps that preserve the determinant of matrices. The study of nonlinear pseudo spectrum preserver problems attracted the attention of a number of authors. Cui et al. [2, Theorem 3.3] characterized maps on $M_n(\mathbb{C})$ that preserve the ϵ-pseudo spectrum of the usual product of matrices. They proved that a map φ on $M_n(\mathbb{C})$ satisfies

$$\sigma_\epsilon(\varphi(T)\varphi(S)) = \sigma_\epsilon(TS) \ (T, S \in M_n(\mathbb{C}))$$

if and only if there exist a scalar $c = \pm 1$ and a unitary matrix $U \in M_n(\mathbb{C})$ such that $\varphi(T) = cUTU^*$ for all $T \in M_n(\mathbb{C})$. This result was extended to the infinite dimensional case by Cui et al. [3, Theorem 4.1], where the authors showed that a surjective map φ on $B(H)$ preserves the ϵ-pseudo spectrum of the product of operators if and only if it is a unitary similarity transform up to a scalar $c = \pm 1$. The aim of this note is to characterize mappings on $B(H)$ that preserve the ϵ-pseudo spectral of the skew Lie product "$[T, S]_* = TS - ST^*$" of operators. For two nonzero vectors x and y in H, let $x \otimes y$ stands for the operator of rank at most one defined by

$$(x \otimes y)z = \langle z, y \rangle \, x, \qquad \forall \ z \in H.$$

Note that every rank one operator in $B(H)$ can be written in this form, and that every finite rank operator $T \in B(H)$ can be written as a finite sum of rank one operators; i.e., $T = \sum_{i=1}^n x_i \otimes y_i$ for some $x_i, y_i \in H$ and $i = 1, 2, ..., n$. We denote by $F(H)$ the set of all finite rank operators in $B(H)$ and $F_n(H)$ the set of all operators of rank at most n, n is a positive integer.

In the following proposition, we collects some known properties of the ϵ-pseudo spectrum which are needed in the proof of the main result.

Let $\epsilon > 0$ be arbitrary and $D(0, \epsilon) = \{\mu \in C : |\mu - a| < \epsilon\}$, where $a \in C$.

Proposition 1.1. *(See [3, 5].)*
Let $\alpha > 0$ and let $T \in B(H)$.
(1) $\sigma(T) + D(0, \epsilon) \subseteq \sigma_\epsilon(T)$.
(2) If T is normal, then $\sigma_\epsilon(T) = \sigma(T) + D(0, \epsilon)$.
(3) For any $\alpha \in \mathbb{C}, \sigma_\epsilon(T + \alpha I) = \alpha + \sigma_\epsilon(T)$.
(4) For any nonzero $\alpha \in \mathbb{C}, \sigma_\epsilon(\alpha T) = \alpha\sigma_{\frac{\epsilon}{|\alpha|}}(T)$.
(5) For any $\alpha \in \mathbb{C}$, we have $\sigma_\epsilon(T) = D(\alpha, \epsilon)$ if and only if $T = \alpha I$.
(6) If $\alpha \in \mathbb{C}$ is a nonzero scalar, then $\sigma_\epsilon(T) = D(0, \epsilon) \cup D(\alpha, \epsilon)$ if and only if there exists a nontrivial orthogonal projection $P \in P(H)$ such that $T = \alpha P$.

2. Main Results

The following lemma is a key tool for the proof of main result and describes the spectrum of the skew Lie product $[x \otimes y, T]_*$ for any nonzero vectors $x, y \in H$ and operator $T \in B(H)$.

Lemma 2.1. *(See [1, Lemma 2.1].) For any nonzero vectors $x, y \in H$ and $T \in B(H)$, set*

$$\Delta_T(x, y) = (\langle Tx, y\rangle + \langle Ty, x\rangle)^2 - 4\|x\|^2 \left\langle T^2y, y\right\rangle$$

and

$$\Lambda_T(x, y) = (\langle x, Ty\rangle + \langle Tx, y\rangle)^2 - 4\|x\|^2 \langle Tx, Ty\rangle$$

Then
(1) $\sigma([x \otimes y, T]_*) = \frac{1}{2}\{0, \langle Tx, y\rangle - \langle Ty, x\rangle \pm \sqrt{\Delta_T(x, y)}\ \}$,
(2) $\sigma([T, x \otimes y]_*) = \frac{1}{2}\{0, \langle Tx, y\rangle - \langle x, Ty\rangle \pm \sqrt{\Lambda_T(x, y)}\ \}$.

Corollary 2.2. *(See [1, Lemma 2.1].) For any $x \in H$ and $T \in B(H)$, we have*

$$\sigma(T(x \otimes x) + (x \otimes x)T) = \{0, \langle Tx, x\rangle \pm \sqrt{\langle T^2x, x\rangle}\ \}.$$

The followig theorem is the main result of this paper.

Theorem 2.3. *Suppose that a surjective map $\varphi : B(H) \to B(H)$ satisfies*

$$\sigma_\epsilon(TS - ST^*) = \sigma_\epsilon(\varphi(T)\varphi(S) - \varphi(S)\varphi(T)^*),\ \ (T, S \in B(H)).$$

Then there exists a unitary operator $U \in B(H)$ such that $\varphi(T) = \mu UTU^*$ for every $T \in B(H)$, where $\mu \in \{-1, 1\}$.

Proof. The proof of it will be completed after checking several claims.

Claim 1. φ is injective and $\varphi(0) = 0$.

Claim 2. φ preserves self-adjoint and anti-self adjoint operators in both directions.

Claim 3. $\varphi(iI) = iI$.

Claim 4. The result in the theorem holds.

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STUDY ON NEW SUBCLASSES OF ANALYTIC FUNCTIONS IN GEOMETRIC FUNCTIONS THEORY

HORMOZ RAHMATAN

Department of Mathematics, Faculty of Science, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran.
h_rahmatan@pnu.ac.ir

Abstract. In this paper, we give characterization of subclasses of analytic functions, we consider $\Phi-$like functions on the unit disc in $\mathbb{C}$ in terms of Löwner chains.

1. Introduction

The class of all analytic functions was denoted by $\mathcal{A}$. Functions belonging to this class can be displayed in the form of the following power series

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n. \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z : z \in \mathbb{C} : |z| < 1\}$. Further, by $\mathcal{S}$. The class of univalent functions in $\mathcal{A}$ which normalized with the conditions $f(0) = f'(0) - 1 = 0$ was represented by $\mathcal{S}$.
Also, let S^* denote the class of starlike functions that is defined as

$$S^* = \left\{ f \in \mathcal{S};\ Re\ \left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \Delta \right\}.$$

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Theorem 1.1. *Let $f : \Delta \to \mathbb{C}$ be a holomorphic function such that $f(0) = 0$ and $f^{'}(0) \neq 0$. Also let $\alpha \in \mathbb{R}$, $|\alpha| < \frac{\pi}{2}$. Then f is spiralike of type α if and only if*

$$Re\ \left(e^{i\alpha} \frac{zf^{'}(z)}{f(z)} \right) > 0,\ \ z \in \Delta$$

Definition 1.2. Let f be analytic in the unit disk Δ of the complex plane with $f(0) = 0, f^{'}(0) \neq 0$. Let Φ be analytic on $f(\Delta)$ with $\Phi = 0$, $Re\, \Phi^{'}(0) > 0$. Then f is Phi-like(in Δ) if

$$Re\ \left(\frac{zf^{'}(z)}{\Phi(f(z))} \right) > 0,\ \ z \in \Delta \tag{1.2}$$

Remark 1.3. The two clasical case of Definition 1.2 are given by $\Phi(\omega) = \omega$(f is starlike) and more generally, $\Phi(\omega) = \lambda\omega$, $Re\, \lambda > 0$(f is apiral-like of type $\arg\lambda$).

Definition 1.4. Let $\mathcal{P}$ denote the class of holomorphic functions p in Δ such that $p(0) = 1$ and Re $p(z) > 0, z \in \Delta$.
This class is usually called the Caratheodory class.

In the lemma and theory of Lowner chains, if f is a function which depends holomorphically on $z \in \Delta$ and is also a function of other real variables, it is customary to write $f'(z, .)$ instead of $\frac{\partial f}{\partial z}(z, .)$.

Lemma 1.5. [1] *The function $f : \Delta \times [0, \infty) \to \mathbb{C}$ with $f(0, t) = 0, f'(0, t) = e^t$, is a Lowner chain if and only if the following conditions hold:*
(i) There exist $r \in (0, 1)$ and a constant $M \geq 0$ such that $f(., t)$ is holomorphic on Δ_r for each $t \geq 0$, where $\Delta = \{z \in \mathbb{C} : |z| < r\}$, locally absolutely continuous in $t \geq 0$ locally uniformly with respect to $z \in \Delta_r$, and

$$|f(z, t)| \leq Me^t, \qquad |z| \leq r, \quad t \geq 0.$$

(ii) There exists a function $p(z, t)$ such that $p(., t) \in \mathcal{P}$ for each $t \geq 0, p(z, .)$ is measurable on $[0, \infty)$ for each $z \in \Delta$, and for all $z \in \Delta_r$,

$$\frac{\partial f}{\partial t}(z, t) = zf'(z, t)p(z, t), \qquad a.e. \quad t \geq 0.$$

(iii) For each $t \geq 0, f(., t)$ is the analytic continuation of $f(., t)|_{\Delta_r}$ to Δ, and furthermore this analytic continuation exists under the assumptions (i) and (ii).

Lemma 1.6. *Let $f(z, t)$ be a Lowner chain. Then there exists a function $p(z, t)$ such that $p(., t) \in \mathcal{P}$, $t \geq 0$, $p(z, t)$ is measurable in $t \in [0, \infty)$ for each $z \in \Delta$, and*

$$\frac{\partial f}{\partial t}(z, t) = zf'(z, t)p(z, t), \quad z \in \Delta, t \geq 0. \tag{1.3}$$

some researchers investigate on subclasses of univalent functions by using of new methods [2, 5, 3, 4].
In this paper, we introduce new subclass of univalent functions, also we shall obtain characterization of the subclass by using the lowner chains method.

2. Main Results

Definition 2.1. Let f be analytic in the unit disk Δ of the complex plane with $f(0) = 0, f^{'}(0) \neq 0$. Let Φ be analytic on $f(\Delta)$ with $\Phi = 0$, $Re\,\Phi^{'}(0) > 0$. Then f is almost Φ-like function of order α(in Δ, $0 \leq \alpha < 1$)if

$$Re\ \left(\frac{\Phi(f(z))}{zf^{'}(z))}\right) > 0,\ z \in \Delta \tag{2.1}$$

Theorem 2.2. *Let f is Φ-like in Δ. Then f is univalent in Δ and $f(\Delta)$ is Φ-like.*

Corollary 2.3. *Let f be analytic in Δ with $f(0) = 0$. Then f is univalent in Δ if and only if f is Φ-like for some Φ.*

Theorem 2.4. *Let D be simply connected subset of $\mathbb{C}$, so suppose $f \in \mathcal{A}$ be a Φ-like function on Δ with $\Phi^{'}(0) = 1$ and let $f(z) \in D$ if and only if $f(z, t) = e^t\Phi(f(z))$ is a lowner chain.*

Theorem 2.5. *Let $f \in \mathcal{A}$ be a Φ-like function on Δ with $\Phi^{'}(0) = 1$, $\omega \in f(\Delta) - 0.$, if and only if $f(z, t) = e^t f(z)$ is a lowner chain.*

Theorem 2.6. *Let D be simply connected subset of $\mathbb{C}$, so let $f \in \mathcal{A}$ be a almost Φ-like function of order α on Δ with $\Phi^{'}(0) = 1$ and let $f(z) \in D$ if and only if*

$$g(z, t) = e^{\frac{1}{1-\alpha}t}\Phi(f(e^{\frac{\alpha}{\alpha-1}t}z)), \quad (0 \leq \alpha < 1,\ z \in \Delta, t \geq 0)$$

is a lowner chain.

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Oral Presentation **$*$: Speaker**

STABILITY RESULTS OF FRACTIONAL DIFFERENTIAL EQUATIONS IN THE HILFER SENSE IN MATRIX-VALUED MENGER SPACES

SAFOURA REZAEI ADERYANI AND REZA SAADATI*

School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran
rsaadati@iust.ac.ir

Abstract. In the present paper, we use some special functions to present the notion of multi-stability and obtain an approximation of fractional differential equations through a fixed point theory. Moreover, some UH stability results for the governing models in different cases are gained.

1. Introduction

Assume the non homogenous vector-valued fractional differential equation given by

$$^{\mathcal{H}}\mathbf{D}^{a,\sigma}\Phi(\lambda) = \theta\Phi(\lambda) + \Phi(\lambda)\overline{\theta} + \Psi(\lambda), \qquad \Phi(0) = \lambda_0, \tag{1.1}$$

in which $^{\mathcal{H}}\mathbf{D}^{a,\sigma}$ is the Hilfer fractional derivative of order a and parameter σ, and $0 < \lambda < \omega < +\infty$. Assume ζ_n be a matrix of n^2.

Consider the following cases:

(1) : $\overline{\theta}, \theta = 0_{1\times 1}, \lambda_0, \Phi, \Psi \in \zeta_1$,
(2) : $\overline{\theta} = 0_{1\times 1}$, $\theta \in \zeta_n$, $\lambda_0, \Phi, \Psi \in \zeta_{n\times 1}$,
(3) : $\overline{\theta} = 0_{m\times m}$, $\theta \in \zeta_n$, $\lambda_0, \Phi, \Psi \in \zeta_{n\times m}$,
(4) : $\overline{\theta} \in \zeta_m$, $\theta \in \zeta_n$, $\lambda_0, \Phi, \Psi \in \zeta_{n\times m}$.

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* Speaker.

In case (1), we use some special functions to study a class of matrix-valued random controllers and also to present the notion of multi-stability. Next, we show the equation (1.1) is the multi-stable. In other cases, via the fixed point theory, we study the UH stability for the equation (1.1).

2. Preliminaries

Assume $\mathcal{O} = [0,1]$ and

$$\mathrm{diag}\zeta_n(\mathcal{O}) = \left\{ \begin{bmatrix} \upsilon_1 & & \\ & \ddots & \\ & & \upsilon_n \end{bmatrix} = \mathrm{diag}[\upsilon_1, \cdots, \upsilon_n],\ \upsilon_1, ..., \upsilon_n \in \mathcal{O} \right\}.$$

We denote $\upsilon := \mathrm{diag}[\upsilon_1, \cdots, \upsilon_n] \preceq \beta := \mathrm{diag}[\beta_1, \cdots, \beta_n]$ if $\upsilon_i \leq \beta_i$ for all $1 \leq i \leq n$.

Next, we define generalized t-norm (GTN) on $\mathrm{diag}\zeta_n(\mathcal{O})$.

Definition 2.1. A GTN on $\mathrm{diag}\zeta_n(\mathcal{O})$ is an operation $\circledast : \mathrm{diag}\zeta_n(\mathcal{O}) \times \mathrm{diag}\zeta_n(\mathcal{O}) \to \mathrm{diag}\zeta_n(\mathcal{O})$ satisfying the conditions below:

(1) $(\forall \upsilon \in \mathrm{diag}\zeta_n(\mathcal{O}))(\upsilon \circledast \mathbf{1}) = \upsilon)$ (boundary condition);

(2) $(\forall (\upsilon, \beta) \in (\mathrm{diag}\zeta_n(\mathcal{O}))^2)(\upsilon \circledast \beta = \beta \circledast \upsilon)$ (commutativity);

(3) $(\forall (\upsilon, \beta, \gamma) \in (\mathrm{diag}\zeta_n(\mathcal{O})^3)(\upsilon \circledast (\beta \circledast \gamma) = (\upsilon \circledast \beta) \circledast \gamma)$ (associativity);

(4) $(\forall (\upsilon, \upsilon', \beta, \beta') \in (\mathrm{diag}\zeta_n(\mathcal{O}^4)(\upsilon \preceq \upsilon'$ and $\beta \preceq \beta' \Longrightarrow \upsilon \circledast \beta \preceq \upsilon' \circledast \beta'$ (monotonicity).

For any $\upsilon, \beta \in \mathrm{diag}\zeta_n(\mathcal{O})$ and any sequences $\{\upsilon_k\}$ and $\{\beta_k\}$ converging to υ and β, if we get $\lim_k(\upsilon_k \circledast \beta_k) = \upsilon \circledast \beta$, thus $\circledast$ on $\mathrm{diag}\zeta_n(\mathcal{O})$ is continuous.

Presume $\mathcal{Z}^+$, the set of matrix distribution functions, including increasing and left continuous maps $\psi : \mathbb{R} \cup \{-\infty, \infty\} \to \mathrm{diag}\zeta_n(\mathcal{O})$ s.t. $\psi_0 = \mathbf{0}$ and $\psi_{+\infty} = \mathbf{1}$. Now $\Delta^+ \subseteq \mathcal{Z}^+$ are all mappings $\psi \in \mathcal{Z}^+$ for which $\ell^- \psi_\varepsilon = \lim_{\sigma \to \varepsilon^-} \psi_\sigma = \mathbf{1}$.

In $\mathcal{Z}^+$, we define "$\preceq$" as: $\Psi \preceq \psi \iff \Psi_\varepsilon \preceq \psi_\varepsilon,\ \forall \varepsilon \in \mathbb{R}$. In addition

$$\nabla_r^j = \begin{cases} \mathbf{0}, & \text{if } r \leq j, \\ \mathbf{1}, & \text{if } r > j \end{cases}$$

belongs to $\mathcal{Z}^+$ and for each matrix distribution function ψ, $\psi \preceq \nabla^0$.

Definition 2.2. Assume $\circledast$ be a continuous GTN, $\mathcal{J}$ be a linear space, and $\psi : \mathcal{J} \to \Delta^+$ be a matrix distribution function. The triple $(\mathcal{J}, \psi, \circledast)$ is called a matrix Menger normed space if we get

(1) $\psi_\varepsilon^j = \nabla_\varepsilon^0$ for all $\varepsilon > 0$ if and only if $j = 0$;

(2) $\psi_\varepsilon^{\nu j} = \psi^j_{\frac{\varepsilon}{|\nu|}}$ for any $s \in \mathcal{J}$ and $\nu \in \mathbb{C}$ with $\nu \neq 0$;

(3) $\psi_{\varepsilon+\varepsilon'}^{j+j'} \succeq \psi_\varepsilon^j \circledast \psi_{\varepsilon'}^{j'}$ for any $j, j' \in \mathcal{J}$ and $\varepsilon, \varepsilon' \geq 0$.

A complete matrix Menger normed space is called a matrix Menger Banach space.

For more details, we refer to [1, 2, 3].

3. Multi-stability for (1.1), when $\overline{\theta}, \theta = 0_{1\times 1}, \lambda_0, \Phi, \Psi \in \zeta_{1,1}$,

Assume the following random controller given by

$$\Lambda\bigg(-\frac{|\lambda|^a}{\Theta\varepsilon}\bigg) = \mathrm{diag}\bigg[{}_0\mathbb{H}_0\bigg(-\frac{|\lambda|^a}{\Theta\varepsilon}\bigg), {}_0\mathbb{H}_1[e_1; -\frac{|\lambda|^a}{\Theta\varepsilon}], {}_2\mathbb{H}_1[d_1, d_2; e_1; -\frac{|\lambda|^a}{\Theta\varepsilon}], \quad (3.1)$$
$${}_1\mathbb{H}_1[d_1; e_1; -\frac{|\lambda|^a}{\Theta\varepsilon}], {}_s\mathbb{H}_r\left[-\frac{|\lambda|^a}{\Theta\varepsilon}\middle|{}^{(d_1,D_1),\dots,(d_s,D_s)}_{(e_1,E_1),\dots,(e_r,E_r)}\right], {}_s^v\mathbb{H}_r^w\left[-\frac{|\lambda|^a}{\Theta\varepsilon}\middle|{}^{(d_j,D_j)_{1,s}}_{(e_j,E_j)_{1,r}}\right]\bigg]$$

where $\Theta > 0$, $\varepsilon \in (0,\infty)$, $0 < a < 1$, and ${}_0\mathbb{H}_0, {}_0\mathbb{H}_1, {}_1\mathbb{H}_1, {}_s\mathbb{H}_r, {}_s^v\mathbb{H}_r^w$ are Exponential function, Mittag–Leffler function, Hypergeometric function, Wright function, Fox–Wright function, and Fox's H–function respectively. for more details see [4].

Notice that the Fox's $\mathbb{H}$-function is defined by

$${}_s^v\mathbb{H}_r^w\left[X\middle|{}^{(d_j,D_j)_{1,s}}_{(e_j,E_j)_{1,r}}\right] := \frac{1}{2\pi i}\int_Z O(Y)X^Y dY, \quad (3.2)$$

where $i^2 = -1$, $X \in \mathbb{C}\backslash\{0\}$, $X^Y = \exp(Y[\log|X| + i\arg(X)])$, $\log|X|$ denotes the natural logarithm of $|X|$ and $\arg(X)$ is not necessarily the principal value. For convenience,

$$O(Y) := \frac{\prod_{j=1}^{v}\Gamma(e_j - E_jY)\prod_{j=1}^{w}\Gamma(1-d_j+D_jY)}{\prod_{j=v+1}^{r}\Gamma(1-e_j+E_jY)\prod_{j=w+1}^{s}\Gamma(d_j-D_jY)},$$

where an empty product is interpreted as 1, and the integers v, w, s, r satisfy the inequalities $0 \le w \le s$ and $1 \le v \le r$. Assume the coefficients

$$D_j > 0 \ (j = 1,\dots,s) \quad \text{and} \quad E_j > 0 \ (j = 1,\dots,r),$$

and the complex parameters

$$d_j \ (j = 1,\dots,s) \quad \text{and} \quad e_j \ (j = 1,\dots,r)$$

are constrained such that no poles of integrand in (3.2) coincide, and Z is a suitable contour of the Mellin-Barnes type (in the complex Y-plane) which separates the poles of one product from the others. Further, if we assume

$$\ell := \sum_{j=1}^{w} D_j - \sum_{j=w+1}^{s} D_j + \sum_{j=1}^{v} E_j - \sum_{j=\mathsf{Q}+1}^{r} E_j > 0,$$

then the integral in (3.2) converges absolutely and defines the $\mathbb{H}$-function, which is analytic in the sector: $|\arg(X)| < \frac{1}{2}\ell\pi$ and with the point $X = 0$ being tacitly excluded. Actually, the $\mathbb{H}$–function makes sense and also defines an analytic function of X when either

$$E := \sum_{j=1}^{s} D_j - \sum_{j=1}^{r} E_j < 0 \quad \text{and} \quad 0 < |X| < \infty,$$

or

$$E = 0 \quad \text{and} \quad 0 < |X| < R := \prod_{j=1}^{s} D_j^{-D_j}\prod_{j=1}^{r} E_j^{E_j}.$$

Definition 3.1. Equation (1.1) has Multi-stability, with respect to $\Lambda(-\frac{|\lambda|^a}{\Theta\varepsilon})$, if there is $\hbar > 0$, s.t for all $\Theta > 0$, and all solution Φ to (3.3), there is a solution Φ' to (1.1), with $\psi_\varepsilon^{\Phi-\Phi'} \succeq \Lambda\left(-\dfrac{|\lambda|^a}{\hbar\Theta\varepsilon}\right)$, where $\varepsilon \in (0, \infty)$.

Theorem 3.2. *Let* (1.1)*, when* $\overline{\theta}, \theta = 0_{1\times 1}, \lambda_0, \Phi, \Psi \in \zeta_{1,1}$*, also Let*

$$\psi_\varepsilon^{{}^{\mathcal{H}}\mathbf{D}^{\sigma,\delta}\Phi(\lambda)-\Psi(\lambda)} \succeq \Lambda\left(-\frac{|\lambda|^a}{\Theta\varepsilon}\right). \tag{3.3}$$

Then (1.1) *is Multi-stable, with respect to* $\Lambda(-\frac{|\lambda|^a}{\Theta\varepsilon})$.

4. UH stability for (1.1), when $\overline{\theta} = 0_{1\times 1}$, $\theta \in \zeta_n$, $\lambda_0, \Phi, \Psi \in \zeta_{n,1}$,

Theorem 4.1. *If each the eigenvalues of* θ *satisfy* $|\arg(\mu(\theta))| > \dfrac{a\pi}{2}$*. Then,* (1.1) *is UH stable.*

5. UH stability for (1.1), when $\overline{\theta} = 0_{m\times m}$, $\theta \in \zeta_n$, $\lambda_0, \Phi, \Psi \in \zeta_{n,m}$

Theorem 5.1. *If any the eigenvalues of* θ *satisfy* $|\arg(\mu(\theta))| > \dfrac{a\pi}{2}$*. Then,* (1.1) *is UH stable.*

6. UH stability for (1.1), when $\overline{\theta} \in \zeta_m$, $\theta \in \zeta_n$, $\lambda_0, \Phi, \Psi \in \zeta_{n,m}$

Theorem 6.1. *Assume all the eigenvalues of* θ *and* $\overline{\theta}$ *satisfy*

$$|\arg(\mu(\theta))| > \frac{a\pi}{2}, \quad \pi \geq |\arg(\mu(\overline{\theta}))| \geq k \quad (\frac{a\pi}{2} < k < \min\{\pi, \pi a\}).$$

then (1.1) *is UH stable.*

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Oral Presentation

HAHN-BANACH THEOREM BEFORE BANACH

REZA REZAVAND

School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran
rezavand@ut.ac.ir

ABSTRACT. In this talk we review the early development of Hahn-Banach theorem before Banach gave the final form of this theorem.

1. INTRODUCTION

One of the most important results in Functional analysis is the Hahn-Banach theorem and its consequences. Before Banach, some of mathematicians had obtained the result for some of special classical spaces such as L^p and $C[a,b]$. In this talk we review their efforts. In translation and rewriting their theorems, I tried to be close to their notations.

2. SCHMIDT AND THE FIRST STEPS

The idea of generalizing finite dimensional euclidean spaces to spaces with infinite dimension was noticed at the beginning of the 20th century. Inspired by Hilbert's works, Schmidt in 1908 made the first systematic study of sequence space ℓ^2 [5]. In that paper Schmidt considered numerical sequences whose sum of squares of the absolute value of their terms is finite. Next he defined inner product (without conjugate on the second component and so to obtain the inner product he used $(A, \overline{B})$), norm and orthogonality as usual and deduced Bessel's equation. Next he introduced convergence in norm (which he named strong convergence: *Starken convergenz*) and

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GramSchmidt process. After these preliminary steps in chapter 2 he considered the system of linear equations with infinite unknowns. After studying homogenous equations, he considered nonhomogenous system of equations $(\overline{A_n}, Z) = c_n$ where for each n, $A_n \in \ell^2$ and $c_n \in \mathbb{C}$ and A_n's are independent. By Gram-Schmidt process, he transformed A_n to orthonormal B_n and the system of equations is transformed to $(\overline{B_n}, Z) = g_n$ and by application of Bessel's equation he proved the following:

Theorem 2.1. *The necessary and sufficient condition for the solvability of the equation is the convergence of the series $\sum |g_n|^2$, in other words $g_n \in \ell^2$.*

In fact, The above problem is related to extension of a functional, because since ${\ell^2}^* = \ell^2$, hence solvability of the above equation is equivalent to existence of $Z \in \ell^2$, such that $Z(\overline{A_n}) = c_n$. But Schmidt did not go further and the next step was taken by Riesz.

3. Riesz and Helly

In 1909, Riesz introduced the spaces $L^p[a, b]$[3] and obtained most of the classical results about these spaces. But the most relevant result to Hahn-Banach theorem that was proved in this paper was:

Theorem 3.1. *A finite or countably infinite system of linear integral equations*

$$\int_a^b f_i(x)\xi(x)dx = c_i, (i = 1, 2, ...)$$

whose coefficient functions $f_i(x)$ belong to $L^{\frac{p}{p-1}}$, has a solution ξ with condition

$$\int_a^b |\xi(x)|^p dx \leq M^p$$

if and only if for each n and each complex numbers μ_i,

$$|\sum_{i=1}^{n} \mu_i c_i|^{\frac{p}{p-1}} \leq M^{\frac{1}{p-1}} \int_a^b |\sum_{i=1}^{n} \mu_i f_i(x)|^{\frac{p}{p-1}} dx.$$

This theorem is the first form of Hahn-Banach theorem in special case $X = L^p[a, b]$ and $X^* = L^q[a, b]$ where $q = \frac{p}{p-1}$ is the conjugate of p.

In 1911, Riesz proved a similar result for $C[a, b]$ [4].

Theorem 3.2. *The system of linear integral equations*

$$\int_a^b f_k(x)d\alpha(x) = c_k, (k = 1, 2, ...)$$

has a solution α if and only if there exists a number M such that for any μ_k

$$|\sum_{k=1}^{n} \mu_k c_k| \leq M \times Max|\sum_{k=1}^{n} \mu_k f_k(x)|.$$

Total variation of the solution α is less than or equal with M.

In 1912 Helly reproved some results of Riesz about $C[a,b]$ [2], specially the above theorem but with a different approach. To obtain the above theorem, Helly first proved a lemma that is similar to modern proof of the Hahn-Banach theorem:

Theorem 3.3. *If for any μ_i's, the inequality*

$$|\sum_{i=1}^{n} \mu_i \gamma_i| \leq M|\sum_{i=1}^{n} \mu_i g_i(x)|$$

holds where for $i = 1, ..n$, γ_i's are fixed real numbers and $g_i \in C[a,b]$, then for each $g_{n+1} \in C[a,b]$, there exists γ_{n+1} such that for any choice of μ_i's, we have

$$|\sum_{i=1}^{n+1} \mu_i \gamma_i| \leq M|\sum_{i=1}^{n+1} \mu_i g_i(x)|.$$

4. Hahn

At last in 1927, Hans Hahn gave the final form of the theorem for real complete normed spaces [1]. But he used norm instead of sublinear functional. In his notation, a linear functional f on a normed space $\Re$ with norm D has slope M, if for each $x \in \Re$,

$$|f(x)| \leq MD(x).$$

Theorem 4.1. *Let $\Re_0$ be a complete linear subspace of $\Re$ and $f_0(x)$ a linear functional on $\Re_0$ of slope M. Then there is a linear functional $f(x)$ on $\Re$ of slope M which coincides with $f_0(x)$ on $\Re_0$.*

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Oral Presentation

ON GENERALIZED FEJÉR INEQUALITY AND A CLASS OF FRACTIONAL INTEGRALS

MOHSEN ROSTAMIAN DELAVAR

Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran
m.rostamian@ub.ac.ir

Abstract. A real mapping $\mathcal{M}_f^{\omega}(t)$ is introduced, a generalized form of Fejér's inequality is obtained and some new and generalized inequalities in connection with fractional integrals and monotone functions are given.

1. Introduction and Preliminaries

Lipót Fejér (1880-1959) in 1906 [4], while studying trigonometric polynomials, discovered the following integral inequalities which later became known as Fejér's inequality (in some references is separated to the left and right):

$$\mathcal{F}\Big(\frac{a+b}{2}\Big)\int_a^b \mathcal{G}(x)dx \le \int_a^b \mathcal{F}(x)\mathcal{G}(x)dx \le \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}\int_a^b \mathcal{G}(x)dx, \quad (1.1)$$

where $\mathcal{F}$ is a convex function ([9]) in the interval (a,b) and $\mathcal{G}$ is a positive function in the same interval such that

$$\mathcal{G}(a+t)=\mathcal{G}(b-t), \qquad 0\le t\le \frac{a+b}{2},$$

i.e., $y=\mathcal{G}(x)$ is a symmetric curve with respect to the straight line which contains the point $(\frac{a+b}{2},0)$ and is normal to the x-axis. In fact the Fejér's

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inequality (1.1), is the weighted version of celebrated Hermite-Hadamard's inequality for convex function $f : [a, b] \to \mathbb{R}$:

$$\mathcal{F}\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b \mathcal{F}(x)dx \leq \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}. \tag{1.2}$$

Our aim in this paper is obtaining a generalized form of Fejér's inequality and applying it to give some new and generalized inequalities in connection with fractional integrals and monotone functions. We introduce a real mapping $\mathcal{M}_f^{\omega}(t)$ and obtain some basic properties for it. Also we use the concept of h-convexity introduced by S. Varošanec in 2006 ([13]):

Definition 1.1. We say that a non-negative function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex or $f \in SX(h, I)$, if for non-negative function $h : (0, 1) \subseteq J \subseteq \mathbb{R} \to \mathbb{R}$ ($h \not\equiv 0$), all $x, y \in I$ and $\alpha \in (0, 1)$ we have

$$\mathcal{F}(\alpha x + (1-\alpha)y) \leq h(\alpha)\mathcal{F}(x) + h(1-\alpha)\mathcal{F}(y).$$

f is said to be h-concave or $f \in SV(h, I)$, If above inequality is reversed.

The mapping $\mathcal{M}_f^{\omega}(t)$. For two real numbers $a < b$, consider integrable functions $f : [a, b] \to \mathbb{R}$ and $\omega : [a, b] \to \mathbb{R}^+ \cup \{0\}$. Define a mapping $\mathcal{M}_f^{\omega}(t) : [0, 1] \to \mathbb{R}$ as

$$\mathcal{M}_f^{\omega}(t) = \int_a^{m_t(\mathcal{L},\mathcal{R})} f(x)\omega(x)dx + \int_{M_t(\mathcal{L},\mathcal{R})}^b f(x)\omega(x)dx,$$

such that

$$m_t(\mathcal{L}, \mathcal{R}) = \min\{\mathcal{L}(t), \mathcal{R}(t)\}, M_t(\mathcal{L}, \mathcal{R}) = \max\{\mathcal{L}(t), \mathcal{R}(t)\}$$

where $\mathcal{L}(t) : [0, 1] \to [a, b]$ and $\mathcal{R}(t) : [0, 1] \to [a, b]$ are considered as the following:

$$\mathcal{L}(t) = tb + (1-t)a, \mathcal{R}(t) = ta + (1-t)b$$

for any $t \in [0, 1]$. Note that

$$\mathcal{M}_f^{1}(t) = \int_a^{m_t(\mathcal{L},\mathcal{R})} f(x)dx + \int_{M_t(\mathcal{L},\mathcal{R})}^b f(x)dx,$$

where by 1, we mean $\omega \equiv 1$.

2. Generalization and refinement of Fejér's Inequality

The following result presents a new and generalized type of the celebrated Fejér's inequality in connection with h-convex functions.

Theorem 2.1. *Consider two integrable functions $f : [a, b] \to \mathbb{R}$ and $w : [a, b] \to \mathbb{R}^+ \cup \{0\}$ such that f is h-convex and ω is symmetric with respect* *to* $\frac{a+b}{2}$. *For all* $t \in [0,1]$, *the following inequality hold:*

$$\frac{1}{2h(\frac{1}{2})} f\Big(\frac{a+b}{2}\Big) \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \omega(x)dx \le \int_a^b f(x)\omega(x)dx - \mathcal{M}_f^{\omega}(t) \tag{2.1}$$
$$\le \frac{|\mathcal{R}(t)-\mathcal{L}(t)|[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{(\mathcal{L}(t)-\mathcal{R}(t))} \int_{\mathcal{R}(t)}^{\mathcal{L}(t)} h\Big(\frac{x-\mathcal{R}(t)}{\mathcal{L}(t)-\mathcal{R}(t)}\Big)\omega(x)dx$$
$$= \frac{|\mathcal{R}(t)-\mathcal{L}(t)|\Big([f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)]\Big)}{(\mathcal{R}(t)-\mathcal{L}(t))} \int_{\mathcal{L}(t)}^{\mathcal{R}(t)} h\Big(\frac{x-\mathcal{L}(t)}{\mathcal{R}(t)-\mathcal{L}(t)}\Big)\omega(x)dx.$$

Inequality (2.1) is a generalization of many Fejér's type inequalities obtained for h-convex functions in literature. However if we set $h(s) = s$ in (2.1), then the following inequality holds:

$$f\Big(\frac{a+b}{2}\Big) \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \omega(x)dx \le \int_a^b f(x)\omega(x)dx - \mathcal{M}_f^{\omega}(t) \tag{2.2}$$
$$\le \frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{|\mathcal{R}(t)-\mathcal{L}(t)|} \int_{\mathcal{L}(t)}^{\mathcal{R}(t)} (x-\mathcal{L}(t))\omega(x)dx$$
$$= \frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{|\mathcal{L}(t)-\mathcal{R}(t)|} \int_{\mathcal{R}(t)}^{\mathcal{L}(t)} (x-\mathcal{R}(t))\omega(x)dx,$$

Inequality (2.2) is a new generalized Fejér's type inequality related to the convex functions.

3. Fractional Integrals

In this section, we introduce a new class of fractional integrals and just consider one special case which is known in literature as Riemann-Liouville fractional integrals (see [5, 7, 8, 10]) to find some hermite-hadamard's type inequalities for it by using generalized Fejér inequality obtained in previous section.
For $t \in [0,1] \setminus \{\frac{1}{2}\}$ consider a bifunction
$G : [m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})] \times [m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})] \to \mathbb{R}^+ \cup \{0\}$ and define the following class of fractional integrals:

$$\mathcal{F}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x) = \int_{m_t(\mathcal{L},\mathcal{R})}^{x} G(x,u)f(u)du, \qquad x > m_t(\mathcal{L},\mathcal{R})$$

and

$$\mathcal{F}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x) = \int_{x}^{M_t(\mathcal{L},\mathcal{R})} G(x,u)f(u)du, \qquad x < M_t(\mathcal{L},\mathcal{R})$$

if above integrals exist.
Now we discuss a special case of $\mathcal{F}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x)$ and $\mathcal{F}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x)$ and obtain some results in connection with Theorem 2.1.

In $\mathcal{F}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x)$ and $\mathcal{F}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x)$ for $\alpha > 0$, consider

$$G(x,u) = \frac{1}{\Gamma(\alpha)}|x-u|^{\alpha-1},\ x,u \in [m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})].$$

So we achieve the following generalized Riemann-Liouville fractional integrals:

$$\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_{m_t(\mathcal{L},\mathcal{R})}^{x}(x-u)^{\alpha-1}f(u)du \qquad x > m_t(\mathcal{L},\mathcal{R})$$

and

$$\mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{M_t(\mathcal{L},\mathcal{R})}(u-x)^{\alpha-1}f(u)dt \qquad M_t(\mathcal{L},\mathcal{R}) < x.$$

Fractional integrals $\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}f(x)$ and $\mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}f(x)$ in special case ($t = 0,1$) reduce to $J^{\alpha}_{a+}f(x)$ and $J^{\alpha}_{b-}f(x)$ respectively, which are known as Riemann-Liouville fractional integrals. Now in Theorem 2.1, consider

$$\omega(x) = \frac{(M_t(\mathcal{L},\mathcal{R})-x)^{\alpha-1} + (x-m_t(\mathcal{L},\mathcal{R}))^{\alpha-1}}{\Gamma(\alpha)},\quad x \in [m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})].$$

Then:

$$\frac{1}{h(\frac{1}{2})}f\Big(\frac{a+b}{2}\Big) \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}|1-2t|^{\alpha}}[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f](M_t(\mathcal{L},\mathcal{R})) + \mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f](m_t(\mathcal{L},\mathcal{R}))] \tag{3.1}$$

$$\leq \alpha[f\circ\mathcal{L}(t) + f\circ\mathcal{R}(t)]\int_0^1 H(s)s^{\alpha-1}ds,$$

for $t \in [0,1]\setminus\{\frac{1}{2}\}$.

In the case that $h(s) = s$, from (3.1) we reach the following inequality which is generalization of inequality (2.1) obtained in [12]:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}|1-2t|^{\alpha}}[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f](M_t(\mathcal{L},\mathcal{R})) + \mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f](m_t(\mathcal{L},\mathcal{R}))]$$
$$\leq \frac{f\circ\mathcal{L}(t) + f\circ\mathcal{R}(t)}{2}.$$

4. Refinements for Hermite-Hadamard's Inequality by Monotone Functions

In this section, we obtain some refinements for Hermite-Hadamard's inequality by the use of fractional integrals discussed in previous section provided that considered functions are nonnegative and monotone. We focus on Riemann-Liouville fractional integrals but results can be extended to many classes of fractional integrals. We need the following result which is a consequence of Theorem 1 in [1](see also [3, 6]).

Theorem 4.1. *If f_1 and f_2 are nonnegative increasing functions on* $[0,1]$, *Then*

$$\int_0^1 f_1(x)dx \int_0^1 f_2(x)dx \leq \int_0^1 f_1(x)f_2(x)dx.$$

Here we give some refinements for Hermite-Hadamard's inequality by the use of fractional integrals for h-convex functions:

Theorem 4.2. *Suppose that $f : [a,b] \to \mathbb{R}$ is an integrable h-convex function and $t \in [0,1] \setminus \{\frac{1}{2}\}$. Then*
(i) For $\alpha \geq 1$, the following inequality holds if f is nonnegative and increasing :

$$\frac{1}{2h(\frac{1}{2})} f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{|1-2t|(b-a)} \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(u)du \tag{4.1}$$
$$\leq \frac{\Gamma(\alpha+1)}{2|1-2t|^{\alpha}(b-a)^{\alpha}} \Big[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f](M_t(\mathcal{L},\mathcal{R})) + \mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f](m_t(\mathcal{L},\mathcal{R}))\Big]$$
$$\leq \alpha \Big[\frac{f \circ \mathcal{L}(t) + f \circ \mathcal{R}(t)}{2}\Big] \int_0^1 H(s)s^{\alpha-1}ds.$$

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APPLICATIONS OF FEJÉR'S INEQUALITY IN CONNECTION WITH EULER'S BETA AND GAMMA FUNCTIONS

MOHSEN ROSTAMIAN DELAVAR

Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran
m.rostamian@ub.ac.ir

Abstract. Some new and generalized results related to the Euler's beta and gamma functions are presented by the use of generalized Fejér's Inequality.

1. Introduction and Preliminaries

Lipót Fejér (1880-1959) in 1906 [4], while studying trigonometric polynomials, discovered the following integral inequalities which later became known as Fejér's inequality (in some references is separated to the left and right):

$$\mathcal{F}\Big(\frac{a+b}{2}\Big)\int_a^b \mathcal{G}(x)dx \leq \int_a^b \mathcal{F}(x)\mathcal{G}(x)dx \leq \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}\int_a^b \mathcal{G}(x)dx, \quad (1.1)$$

where $\mathcal{F}$ is a convex function ([6]) in the interval (a,b) and $\mathcal{G}$ is a positive function in the same interval such that

$$\mathcal{G}(a+t)=\mathcal{G}(b-t), \qquad 0 \leq t \leq \frac{a+b}{2},$$

i.e., $y=\mathcal{G}(x)$ is a symmetric curve with respect to the straight line which contains the point $(\frac{a+b}{2},0)$ and is normal to the x-axis. In fact the Fejér's

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inequality (1.1), is the weighted version of celebrated Hermite-Hadamard's inequality for convex function $f : [a, b] \to \mathbb{R}$:

$$\mathcal{F}\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a} \int_a^b \mathcal{F}(x)dx \leq \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2}. \tag{1.2}$$

In this paper some new and generalized results related to the Euler's beta and gamma functions are presented by the use of generalized Fejér's Inequality. Also we use the concept of h-convexity introduced by S. Varošanec in 2006 ([9]):

Definition 1.1. We say that a non-negative function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex or $f \in SX(h, I)$, if for non-negative function $h : (0, 1) \subseteq J \subseteq \mathbb{R} \to \mathbb{R}$ $(h \not\equiv 0)$, all $x, y \in I$ and $\alpha \in (0, 1)$ we have

$$\mathcal{F}(\alpha x + (1-\alpha)y) \leq h(\alpha)\mathcal{F}(x) + h(1-\alpha)\mathcal{F}(y).$$

f is said to be h-concave or $f \in SV(h, I)$, If above inequality is reversed.

The following result presents a new and generalized type of the celebrated Fejér's inequality in connection with h-convex functions.

Theorem 1.2. *Consider two integrable functions $f : [a, b] \to \mathbb{R}$ and $w : [a, b] \to \mathbb{R}^+ \cup \{0\}$ such that f is h-convex and ω is symmetric with respect to $\frac{a+b}{2}$. For all $t \in [0, 1]$, the following inequality hold:*

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})} f\Big(\frac{a+b}{2}\Big) \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \omega(x)dx \leq \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(x)\omega(x)dx \qquad (1.3)\\ &\leq \frac{|\mathcal{R}(t) - \mathcal{L}(t)|[f \circ \mathcal{L}](t) + [f \circ \mathcal{R}](t)}{(\mathcal{L}(t) - \mathcal{R}(t))} \int_{\mathcal{R}(t)}^{\mathcal{L}(t)} h\Big(\frac{x - \mathcal{R}(t)}{\mathcal{L}(t) - \mathcal{R}(t)}\Big)\omega(x)dx \\ &= \frac{|\mathcal{R}(t) - \mathcal{L}(t)|\Big([f \circ \mathcal{L}](t) + [f \circ \mathcal{R}](t)]\Big)}{(\mathcal{R}(t) - \mathcal{L}(t))} \int_{\mathcal{L}(t)}^{\mathcal{R}(t)} h\Big(\frac{x - \mathcal{L}(t)}{\mathcal{R}(t) - \mathcal{L}(t)}\Big)\omega(x)dx, \end{aligned}$$

where

$$m_t(\mathcal{L}, \mathcal{R}) = \min\{\mathcal{L}(t), \mathcal{R}(t)\}, M_t(\mathcal{L}, \mathcal{R}) = \max\{\mathcal{L}(t), \mathcal{R}(t)\}$$

and $\mathcal{L}(t) : [0, 1] \to [a, b]$, $\mathcal{R}(t) : [0, 1] \to [a, b]$ are considered as the following:

$$\mathcal{L}(t) = tb + (1-t)a, \mathcal{R}(t) = ta + (1-t)b$$

for any $t \in [0, 1]$.

Inequality (1.3) is a generalization of many Fejér's type inequalities obtained for h-convex functions in literature. However if we set $h(s) = s$ in

(1.3), then the following inequality holds:

$$f\Big(\frac{a+b}{2}\Big)\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)dx \le \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(x)\omega(x)dx \tag{1.4}$$
$$\le \frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{|\mathcal{R}(t)-\mathcal{L}(t)|}\int_{\mathcal{L}(t)}^{\mathcal{R}(t)}(x-\mathcal{L}(t))\omega(x)dx$$
$$= \frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{|\mathcal{L}(t)-\mathcal{R}(t)|}\int_{\mathcal{R}(t)}^{\mathcal{L}(t)}(x-\mathcal{R}(t))\omega(x)dx,$$

2. Gamma and Beta Function

In this section, we present some inequalities and results related to gamma and beta functions. Specially by considering appropriate functions in Theorem 1.2 along with some calculations, we give a simple proof for well known Stirling's formula as well.
The Euler's integral of the second kind i.e. Gamma function [2] is defined as:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \qquad Re(x) > 0.$$

Consider the function $f(x) = \ln\Gamma(x), x \in (0,+\infty)$ which is convex ($\Gamma(x)$ is log-convex). Now in Theorem 1.2, consider $h(s) = s$, $t = 0, 1$, $b = a + 1$ for $a \in (0,+\infty)$ and symmetric function $\omega : [a, a+1] \to (0,+\infty)$ with respect to $a + \frac{1}{2}$. Then we obtain the following inequality:

$$\Gamma(a+\frac{1}{2}) \le \exp\Big(\frac{1}{\mathcal{K}}\int_a^{a+1}\omega(x)\ln\Gamma(x)dx\Big) \le \sqrt{\Gamma(a)\Gamma(a+1)}, \tag{2.1}$$

where $\mathcal{K} = \int_a^{a+1}\omega(x)dx$. In special case for $\omega \equiv 1$, by the Raabe's formula [5], i.e.

$$\int_a^{a+1}\ln\Gamma(x)dx = \ln\sqrt{2\pi} + a\ln(a) - a,$$

and inequality (2.1) we have

$$\Gamma(a+\frac{1}{2}) \le \sqrt{2\pi}\big(\frac{a}{e}\big)^a \le \sqrt{\Gamma(a)\Gamma(a+1)}, \tag{2.2}$$

for any $a \in (0,+\infty)$. By applying Wendel's inequality ([10]), i.e.

$$\Big(\frac{a}{a+s}\Big)^{1-s} \le \frac{\Gamma(a+s)}{a^s\Gamma(a)} \le 1,$$

in (2.2) for $s = \frac{1}{2}$, we get to

$$\sqrt{\frac{a}{a+\frac{1}{2}}} \le \frac{\Gamma(a+\frac{1}{2})}{a^{\frac{1}{2}}\Gamma(a)} \le \frac{\sqrt{2\pi a}\big(\frac{a}{e}\big)^r}{\Gamma(a+1)} \le 1. \tag{2.3}$$

So two results can be extracted from inequality (2.3) by using squeeze theorem [7]. The First is

$$\lim_{a\to\infty}\frac{\Gamma(a+\frac{1}{2})}{a^{\frac{1}{2}}\Gamma(a)}=1,$$

and the second is generalization of Stirling's formula [3],

$$\Gamma(a+1)\approx\sqrt{2\pi a}\Big(\frac{a}{e}\Big)^{a}\qquad as\ \ a\to\infty.$$

For the case that $a\in\mathbb{N}$, we recapture the classic Stirling's formula:

$$a!\approx\sqrt{2\pi a}\Big(\frac{a}{e}\Big)^{a}\qquad as\ \ a\to\infty.$$

The Euler's integral of the first kind is known as beta function [1]:

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\qquad Re(x)>0, Re(y)>0.$$

To obtain some results in connection with beta function by the use of Fejér's inequality consider

$$\begin{cases} f(x)=\big(x-m_t(\mathcal{L},\mathcal{R})\big)^{r},\ 0<m_t(\mathcal{L},\mathcal{R})\le x\le M_t(\mathcal{L},\mathcal{R}), r\in[1,\infty);\\[2ex] \omega(x)=\dfrac{\big(M_t(\mathcal{L},\mathcal{R})-x\big)^{P-1}\big(x-m_t(\mathcal{L},\mathcal{R})\big)^{P-1}}{(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R}))^{p}},\ 0<m_t(\mathcal{L},\mathcal{R})\le x\le M_t(\mathcal{L},\mathcal{R});\\[2ex] h(s)=s^{k},\ 0\le k\le 1, s>0, \end{cases}$$

where $0<a<b$, $p>0$ and $t\in[0,1]\setminus\{\frac{1}{2}\}$. It follows with some calculations and Theorem 1.2 that

$$\begin{aligned}&\frac{1}{2(\frac{1}{2})^{k}}.\frac{\big(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})\big)^{r}}{2^{r}}\big(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})\big)^{p-1}\beta(p,p)\\ &\le\big(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})\big)^{p+r-1}\beta(p,p+r)\\ &\le\big(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})\big)^{r+1}\big(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})\big)^{p-2}\beta(k+p,p),\end{aligned}$$

which implies the following inequalities related to beta function:

$$2^{k-r-1}\beta(p,p)\le t\beta(p,p+r)+(1-t)2^{k-r-1}\beta(p,p)\le\beta(p,p+r)\le\beta(k+p,p),\tag{2.4}$$

for $t\in[0,1]\setminus\{\frac{1}{2}\}$, $0\le k\le 1$ and $r\in[1,\infty)$.

Remark 2.1. For the case that $f(x)=\big(M_t(\mathcal{L},\mathcal{R})-x\big)^{r}$, with the same argument as above we recapture (2.4) because of the fact $\beta(p,p+r)=\beta(p+r,p)$.

In special case if we set $k=1$ and $t=0,1$, we get

$$\frac{1}{2^{r}}\beta(p,p)\le\beta(p+r,p)\le\beta(1+p,p)=\frac{1}{2}\beta(p,p),\tag{2.5}$$

for $p > 0$ and $r \in [1, \infty)$. From (2.5) and the characterization $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ we obtain that

$$\frac{1}{2^r} \leq \frac{\Gamma(2p)\Gamma(p+r)}{\Gamma(p)\Gamma(2p+r)} \leq \frac{1}{2},$$

for $p > 0$ and $r \in [1, \infty)$. In more special case for any $p > 0$, we have the following result:

$$\frac{1}{2}\Gamma(p)\Gamma(2p+1) = \Gamma(2p)\Gamma(p+1).$$

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Oral Presentation $*$: **Speaker**

ON OPTIMALITY CONDITIONS FOR COMPOSITE UNCERTAIN MULTIOBJECTIVE OPTIMIZATION

MARYAM SAADATI* AND MORTEZA OVEISIHA

Department of Pure Mathematics, Imam Khomeini International University, Qazvin 34194, Iran
m.saadati@edu.ikiu.ac.ir; oveisiha@sci.ikiu.ac.ir

Abstract. This article is concerned with a nonsmooth/nonconvex composite multiobjective optimization problem involving uncertain constraints in arbitrary Asplund spaces. We first establish necessary optimality conditions for weakly robust efficient solutions of the problem in terms of the limiting subdifferential. Then, sufficient conditions for the existence of (weakly) robust efficient solutions to such a problem are driven under the new concept of pseudo-quasi convexity for composite functions.

1. Introduction

Robust optimization approach considers the cases in which optimization problems often deal with uncertain data due to prediction errors, lack of information, fluctuations, or disturbances. Particularly, in most cases these problems depend on conflicting goals due to multiobjective decision makers which have different optimization criteria. So, the *robust multiobjective optimization* is highly interesting in optimization theory and important in applications. To the best of our knowledge, the most powerful results in this direction were established for finite-dimensional problems not dealing with composite functions. So, an infinite-dimensional framework would be proper to study when involving optimality and duality in composite optimization. From this, our main purpose in this paper is to investigate a

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nonsmooth/nonconvex multiobjective optimization problem with composition fields over arbitrary *Asplund* spaces.

Throughout this paper, we use standard notation of variational analysis; see, for example, [1]. Unless otherwise stated, all the spaces under consideration are *Asplund* with the norm $\|\cdot\|$ and the canonical pairing $\langle\cdot\,,\cdot\rangle$ between the space X in question and its *dual* X^* equipped with the *weak** *topology* w^*. For a given nonempty set $\Omega \subset X$, the symbols $\operatorname{co}\Omega$, $\operatorname{cl}\Omega$, and $\operatorname{int}\Omega$ indicate the *convex hull*, *topological closure*, and *topological interior* of Ω, respectively, while $\operatorname{cl}^*\Omega$ stands for the *weak** *topological closure* of $\Omega \subset X^*$. The *dual cone* of Ω is the set

$$\Omega^+ := \{x^* \in X^* \mid \langle x^*, x\rangle \geq 0,\ \ \forall x \in \Omega\}.$$

Besides, $\mathbb{R}^n_+$ signifies the nonnegative orthant of $\mathbb{R}^n$ for $n \in \mathbb{N} := \{1, 2, \dots\}$.

Suppose that $F : X \to W$ and $f : W \to Y$ be vector-valued functions between Asplund spaces, and that $K \subset Y$ be a pointed (i.e., $K \bigcap (-K) = \{0\}$) closed convex cone. We consider a *composite multiobjective optimization* problem:

$$\begin{aligned} &\text{(CP)} \qquad \min\nolimits_K\ \ (f \circ F)(x) \\ &\qquad\qquad\qquad \text{s.t.}\ \ (g_i \circ G_i)(x) \leq 0, \quad i = 1, 2, \dots, n, \end{aligned}$$

where the functions $G = (G_1, G_2, \dots, G_n) : X \to Z$ and $g = (g_1, g_2, \dots, g_n) : Z \to \mathbb{R}^n$ define the constraints on Asplund spaces. This problem in the face of data uncertainty in the constraints can be captured by the following *composite uncertain multiobjective optimization* problem:

$$\begin{aligned} &\text{(CUP)} \qquad \min\nolimits_K\ \ (f \circ F)(x) \\ &\qquad\qquad\qquad \text{s.t.}\ \ (g_i \circ G_i)(x, v_i) \leq 0, \quad i = 1, 2, \dots, n, \end{aligned}$$

where $x \in X$ is the vector of *decision* variable, v_i's are *uncertain* parameters and $v_i \in \mathcal{V}_i$ for some *sequentially compact topological* space $\mathcal{V}_i$, $\mathcal{V} := \prod_{i=1}^{n} \mathcal{V}_i$, and $G_i : X \times \mathcal{V}_i \to Z \times \mathcal{U}_i$ and $g_i : Z \times \mathcal{U}_i \to \mathbb{R}$, $i = 1, 2, \dots, n$, are given functions for topological spaces $\mathcal{U}_i$, $\mathcal{U} := \prod_{i=1}^{n} \mathcal{U}_i$.

For investigating the problem (CUP), we associate with it the so-called *robust* counterpart:

$$\begin{aligned} &\text{(CRP)} \qquad \min\nolimits_K\ \ (f \circ F)(x) \\ &\qquad\qquad\qquad \text{s.t.}\ \ (g_i \circ G_i)(x, v_i) \leq 0, \quad \forall v_i \in \mathcal{V}_i,\ \ i = 1, 2, \dots, n. \end{aligned}$$

A vector $x \in X$ is called a *robust feasible solution* of problem (CUP) if it is a *feasible solution* of problem (CRP). The *feasible set* C of problem (CRP) is defined by

$$C := \{x \in X \mid (g_i \circ G_i)(x, v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ \ i = 1, 2, \dots, n\}.$$

Definition 1.1. (i) We say that a vector $\bar{x} \in X$ is a *robust efficient solution* of problem (CUP), denoted by $\bar{x} \in \mathcal{S}(\text{CRP})$, if $\bar{x}$ is an *efficient solution* of problem (CRP), i.e., $\bar{x} \in C$ and

$$(f \circ F)(x) - (f \circ F)(\bar{x}) \notin -K \setminus \{0\}, \quad \forall x \in C.$$

(ii) A vector $\bar{x} \in X$ is called a *weakly robust efficient solution* of problem (CUP), denoted by $\bar{x} \in \mathcal{S}^w(\text{CRP})$, if $\bar{x}$ is a *weakly efficient solution* of problem (CRP), i.e., $\bar{x} \in C$ and

$$(f \circ F)(x) - (f \circ F)(\bar{x}) \notin -\operatorname{int} K, \quad \forall x \in C.$$

Motivated by the concept of pseudo-quasi generalized convexity in [4], we introduce a similar concept of pseudo-quasi convexity type for the compositions $f \circ F$ and $g \circ G$ to establish sufficient optimality conditions for (weakly) robust efficient solutions of problem (CUP).

Definition 1.2. (i) We say that $(f \circ F, g \circ G)$ is *type I pseudo convex* at $\bar{x} \in X$ if for any $x \in X$, $y^* \in K^+$, $w^* \in \partial\langle y^*, f\rangle(F(\bar{x}))$, $x^* \in \partial\langle w^*, F\rangle(\bar{x})$, $v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))$, and $x_i^* \in \partial_x\langle v_i^*, G_i\rangle(\bar{x}, v_i)$, $v_i \in \mathcal{V}_i(\bar{x})$, $i = 1, 2, \ldots, n$, there exists $\nu \in X$ such that

$$\begin{aligned} &\langle y^*, f \circ F\rangle(x) < \langle y^*, f \circ F\rangle(\bar{x}) \Longrightarrow \langle x^*, \nu\rangle < 0, \\ &(g_i \circ G_i)(x, v_i) \le (g_i \circ G_i)(\bar{x}, v_i) \Longrightarrow \langle x_i^*, \nu\rangle \le 0, \quad i = 1, 2, \ldots, n. \end{aligned}$$

(ii) We say that $(f \circ F, g \circ G)$ is *type II pseudo convex* at $\bar{x} \in X$ if for any $x \in X \setminus \{\bar{x}\}$, $y^* \in K^+ \setminus \{0\}$, $w^* \in \partial\langle y^*, f\rangle(F(\bar{x}))$, $x^* \in \partial\langle w^*, F\rangle(\bar{x})$, $v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))$, and $x_i^* \in \partial_x\langle v_i^*, G_i\rangle(\bar{x}, v_i)$, $v_i \in \mathcal{V}_i(\bar{x})$, $i = 1, 2, \ldots, n$, there exists $\nu \in X$ such that

$$\begin{aligned} &\langle y^*, f \circ F\rangle(x) \le \langle y^*, f \circ F\rangle(\bar{x}) \Longrightarrow \langle x^*, \nu\rangle < 0, \\ &(g_i \circ G_i)(x, v_i) \le (g_i \circ G_i)(\bar{x}, v_i) \Longrightarrow \langle x_i^*, \nu\rangle \le 0, \quad i = 1, 2, \ldots, n. \end{aligned}$$

Let $\Omega \subset X$ be *locally closed* around $\bar{x} \in \Omega$, i.e., there is a neighborhood U of $\bar{x}$ for which $\Omega \bigcap \operatorname{cl} U$ is closed. The *Fréchet normal cone* $\widehat{N}(\bar{x}; \Omega)$ and the *Mordukhovich normal cone* $N(\bar{x}; \Omega)$ to Ω at $\bar{x} \in \Omega$ are defined by

$$\begin{aligned} &\widehat{N}(\bar{x}; \Omega) := \{x^* \in X^* \mid \limsup_{x \to \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \le 0\}, \\ &N(\bar{x}; \Omega) := \operatorname*{Lim\,sup}_{x \to \bar{x}} \widehat{N}(x; \Omega), \end{aligned}$$

where $x \xrightarrow{\Omega} \bar{x}$ stands for $x \to \bar{x}$ with $x \in \Omega$. If $\bar{x} \notin \Omega$, we put $\widehat{N}(\bar{x}; \Omega) = N(\bar{x}; \Omega) := \emptyset$.

For an extended real-valued function $\phi : X \to \overline{\mathbb{R}}$, the *limiting/Mordukhovich subdifferential* and the *regular/Fréchet subdifferential* of ϕ at $\bar{x} \in \operatorname{dom} \phi$ are given, respectively, by

$$\partial\phi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \phi(x)); \operatorname{epi} \phi)\}$$

and

$$\widehat{\partial}\phi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}((\bar{x}, \phi(x)); \operatorname{epi} \phi)\}.$$

If $|\phi(\bar{x})| = \infty$, then one puts $\partial\phi(\bar{x}) = \widehat{\partial}\phi(\bar{x}) := \emptyset$.

Throughout this paper, we assume that the following assumptions hold:

Assumption 1.3. (See [2, p.131])

(A1) For a fixed $\bar{x} \in X$, F is locally Lipschitz at $\bar{x}$ and f is locally Lipschitz at $F(\bar{x})$.

(A2) For each $i = 1, 2, \ldots, n$, G_i is locally Lipschitz at $\bar{x}$ and uniformly on $\mathcal{V}_i$, and g_i is Lipschitz continuous on $G_i(\bar{x}, \mathcal{V}_i)$.

(A3) For each $i = 1, 2, \ldots, n$, the functions $v_i \in \mathcal{V}_i \mapsto G_i(\bar{x}, v_i) \in Z \times \mathcal{U}_i$ and $G_i(\bar{x}, v_i) \mapsto g_i(G_i(\bar{x}, v_i)) \in \mathbb{R}$ are locally Lipschitzian.

(A4) For each $i = 1, 2, \ldots, n$, we define real-valued functions ϕ_i and ϕ on X via

$$\phi_i(x) := \max_{v_i \in \mathcal{V}_i} (g_i \circ G_i)(x, v_i) \qquad \text{and} \qquad \phi(x) := \max_{i \in \{1,2,\ldots,n\}} \phi_i(x),$$

and we notice that above assumptions imply that ϕ_i is well defined on $\mathcal{V}_i$. In addition, ϕ_i and ϕ follow readily that are locally Lipschitz at $\bar{x}$, since each $(g_i \circ G_i)(\bar{x}, v_i)$ is (see [2, (H1), p.131] and [3, p.290]).

(A5) For each $i = 1, 2, \ldots, n$, the multifunction $(x, v_i) \in X \times \mathcal{V}_i \rightrightarrows \partial_x (g_i \circ G_i)(x, v_i) \subset X^*$ is weak* closed at $(\bar{x}, \bar{v}_i)$ for each $\bar{v}_i \in \mathcal{V}_i(\bar{x})$, where $\mathcal{V}_i(\bar{x}) = \{v_i \in \mathcal{V}_i \mid (g_i \circ G_i)(\bar{x}, v_i) = \phi_i(\bar{x})\}$.

In the rest of this section, we present a suitable constraint qualification in the sense of robustness, which is needed to get a so-called *robust Karush-Kuhn-Tucker* (KKT) *condition.*

Definition 1.4. (See [4, Definition 2.3]) Let $\bar{x} \in C$. We say that the *constraint qualification* (CQ) *condition* is satisfied at $\bar{x}$ if

$$0 \notin \mathrm{cl}^* \mathrm{co}(\cup\{\cup_{v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))} \partial_x \langle v_i^*, G_i \rangle (\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x})\}), \quad i \in I(\bar{x}),$$

where $I(\bar{x}) := \{i \in \{1, 2, \ldots, n\} \mid \phi_i(\bar{x}) = \phi(\bar{x})\}$.

It is worth to mention here that this condition (CQ) is reduced to the *extended Mangasarian-Fromovitz constraint qualification* (EMFCQ) in the *smooth* setting; see e.g., [1] for more details.

Definition 1.5. A point $\bar{x} \in C$ is said to satisfy the *robust* (KKT) *condition* if there exist $y^* \in K^+ \setminus \{0\}$, $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_+$, and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, 2, \ldots, n$, such that

$$0 \in \cup_{w^* \in \partial \langle y^*, f \rangle (F(\bar{x}))} \partial \langle w^*, F \rangle (\bar{x}) + \sum_{i=1}^{n} \mu_i \, \mathrm{cl}^* \mathrm{co}(\cup\{\cup_{v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))} \partial_x \langle v_i^*, G_i \rangle (\bar{x}, v_i) \mid v_i \in \mathcal{V}_i(\bar{x})\}),$$

$$\mu_i \max_{v_i \in \mathcal{V}_i} (g_i \circ G_i)(\bar{x}, v_i) = \mu_i \, (g_i \circ G_i)(\bar{x}, \bar{v}_i) = 0, \quad i = 1, 2, \ldots, n.$$

Therefore, the robust (KKT) condition defined above is guaranteed by the constraint qualification (CQ).

2. Robust necessary and sufficient optimality conditions

The first theorem establishes a necessary optimality condition in the sense of the limiting subdifferential for weakly robust efficient solutions of problem (CUP).

Theorem 2.1. *Suppose that $\bar{x} \in \mathcal{S}^w$(CRP). Then there exist $y^* \in K^+$, $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_+$, with $\|y^*\| + \|\mu\| = 1$, and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, 2, \ldots, n$, such that*

$$\begin{cases} 0 \in \cup_{w^* \in \partial \langle y^*, f\rangle (F(\bar{x}))} \partial \langle w^*, F\rangle(\bar{x}) + \sum_{i=1}^{n} \mu_i \operatorname{cl}^* \operatorname{co}(\cup \{ \cup_{v_i^* \in \partial_x g_i(G_i(\bar{x}, v_i))} \partial_x \langle v_i^*, G_i\rangle(\bar{x}, v_i) \\ \qquad \mid \ v_i \in \mathcal{V}_i(\bar{x})\}), \\ \mu_i \max_{v_i \in \mathcal{V}_i} g_i(G_i(\bar{x}, v_i)) = \mu_i \, g_i(G_i(\bar{x}, \bar{v}_i)) = 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{2.1}$$

Furthermore, if the (CQ) *is satisfied at $\bar{x}$, then* (2.1) *holds with $y^* \neq 0$.*

Remark 2.2. Theorem 2.1 reduces to [4, Theorem 3.2] for the problem (UP), and [5, Proposition 3.9] and [2, Theorem 3.3] in the case of finite-dimensional multiobjective optimization. Note further that our approach here, which involves the fuzzy necessary optimality condition in the sense of the Fréchet subdifferential and the inclusion formula for the limiting subdifferential of maximum functions in the setting of Asplund spaces, is totally different from the last two presented in the aforementioned papers.

The forthcoming theorem presents a (KKT) sufficient optimality conditions for (weakly) robust efficient solutions of problem (CUP).

Theorem 2.3. *Assume that $\bar{x} \in C$ satisfies the robust* (KKT) *condition.*

(i) *If $(f \circ F, g \circ G)$ is type I pseudo convex at $\bar{x}$, then $\bar{x} \in \mathcal{S}^w$(CRP).*

(ii) *If $(f \circ F, g \circ G)$ is type II pseudo convex at $\bar{x}$, then $\bar{x} \in \mathcal{S}$(CRP).*

Remark 2.4. Theorem 2.3 reduces to [4, Theorem 3.4] and [5, Theorem 3.10], and develops [2, Theorem 3.11] and [6, Theorem 3.2] under pseudo-quasi convexity assumptions.

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Oral Presentation **$*$: Speaker**

BOUNDEDNESS OF COMPOSITION OPERATORS IN POLYDISK

Z. SAEIDIKIA* AND A. ABKAR

Department of Pure Mathematics, Imam Khomeini International University, Qazvin 34194, Iran
z.saeidikia@edu.ikiu.ac.ir; abkar@sci.ikiu.ac.ir

Abstract. In this paper we investigate conditions on the symbol function to guarantee that the composition operator from the Bergman space of the polydisk to the Bergman space of the unit disk is bounded.

1. introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. For $\alpha > -1$, the weighted Bergman space $A^2_\alpha(\mathbb{D})$ is the space of analytic functions f in $\mathbb{D}$ for which

$$\int_{\mathbb{D}} |f(z)|^2 \, dA_\alpha(z) < +\infty,$$

where

$$dA_\alpha(z) = \pi^{-1}(\alpha+1)(1-|z|^2)^\alpha dx\, dy$$

is the weighted area measure in the unit disk. It is well-known that $A^2_\alpha(\mathbb{D})$ equipped with the inner product

$$\langle f, g\rangle = (\alpha+1)\int_{\mathbb{D}} f(z)\overline{g(z)}(1-|z|^2)^\alpha dA(z),$$

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* Speaker.

is a Hilbert space with the following reproducing kernel (see [4])

$$K_w(z) = \frac{1}{(1 - z\overline{w})^{\alpha+2}}.$$

We mean by polydisk the subset $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$ of the n-dimensional complex space. Now let $\mathrm{Hol}(\mathbb{D}^n)$ denote the space of holomorphic functions on $\mathbb{D}^n$. The weighted Bergman space on the polydisk $\mathbb{D}^n$ is defined by

$$A^2_\alpha(\mathbb{D}^n) = \mathrm{Hol}(\mathbb{D}^n) \cap L^2(\mathbb{D}^n, dV_\alpha)$$

where

$$dV_\alpha(z) = dA_\alpha(z_1) \cdots dA_\alpha(z_n),$$

and

$$dA_\alpha(z_k) = \pi^{-1}(\alpha + 1)(1 - |z_k|^2)^\alpha dx_k dy_k, \quad 1 \leq k \leq n.$$

This means that a function $f(z_1, ..., z_n)$ in $\mathrm{Hol}(\mathbb{D}^n)$ belongs to $A^2_\alpha(\mathbb{D}^n)$ if

$$\|f\|^2_{A^2_\alpha(\mathbb{D}^n)} = \int_{\mathbb{D}^n} |f(z_1, ..., z_n)|^2 dA_\alpha(z_1) \cdots dA_\alpha(z_n) < +\infty.$$

The reproducing kernel of $A^2_\alpha(\mathbb{D}^n)$ is given by (see the papers [5] and [6])

$$K_z(w) = \prod_{j=1}^{n} \frac{1}{(1 - \overline{z_j} w_j)^{\alpha+2}} = K_{z_1}(w_1) \cdots K_{z_n}(w_n).$$

Let $\Phi : \mathbb{D}^m \to \mathbb{D}^n$ be a holomorphic mapping (m, n are positive integers):

$$\Phi(z) = \big(\varphi_1(z), \ldots, \varphi_n(z)\big), \quad z = (z_1, \ldots, z_m) \in \mathbb{D}^m.$$

Consider the composition operator

$$C_\Phi : A^2_\alpha(\mathbb{D}^n) \to A^2_\beta(\mathbb{D}^m),$$

defined by $C_\Phi(f) = f \circ \Phi$. Moreover, if $\psi : \mathbb{D}^n \to \mathbb{C}$ is holomorphic, then the weighted composition operator $C_{\psi,\Phi}$ is defined by

$$C_{\psi,\Phi}(f) = \psi \cdot f \circ \Phi, \quad f \in A^2_\alpha(\mathbb{D}^n).$$

In this paper, we shall focus on the composition operator

$$C_\Phi : A^2_\alpha(\mathbb{D}^2) \to A^2_\alpha(\mathbb{D}).$$

This problem can then be studied for $C_\Phi : A^2_\alpha(\mathbb{D}^k) \to A^2_\alpha(\mathbb{D})$, and our choice $k = 2$ is for simplicity. We shall prove that if φ and ψ are analytic self mappings of the unit disk, and $\Phi = (\varphi, \psi) : \mathbb{D} \to \mathbb{D}^2$ is a holomorphic function such that $\|\varphi\psi\|_\infty < 1$, then $C_\Phi : A^2_\alpha(\mathbb{D}^2) \to A^2_\alpha(\mathbb{D})$ is bounded. We should mention that this problem for the Hardy space is already known; see [3]. For recent work on this topic see the papers [1] and [2].

2. Preliminaries

Let $F(z,w) \in A^2_\alpha(\mathbb{D}^2)$. Note that

$$C_\Phi F(z) = F(\varphi(z), \psi(z)).$$

Let

$$F(z,w) = \sum_{n=0}^{\infty} z^n F_n(w) = \sum_{n=0}^{\infty} w^n G_n(z).$$

Since for $m \neq n$ we have

$$\int_{\mathbb{D}^2} z^n \overline{z}^m F_n(w)\overline{F_m(w)} dA_\alpha(z) dA_\alpha(w) = 0,$$

it follows that

$$\begin{aligned} \|F\|^2_{A^2_\alpha(\mathbb{D}^2)} &= \sum_{n=0}^{\infty} \|z^n\|^2_{A^2_\alpha(\mathbb{D})} \|F_n\|^2_{A^2_\alpha(\mathbb{D})} \\ &= \sum_{n=0}^{\infty} \frac{n!\,\Gamma(\alpha+2)}{\Gamma(\alpha+n+2)} \|F_n\|^2_{A^2_\alpha(\mathbb{D})}. \end{aligned}$$

Similarly, we see that

$$\|F\|^2_{A^2_\alpha(\mathbb{D}^2)} = \sum_{n=0}^{\infty} \frac{n!\,\Gamma(\alpha+2)}{\Gamma(\alpha+n+2)} \|G_n\|^2_{A^2_\alpha(\mathbb{D})}.$$

Now let σ be a number satisfying

$$\|\varphi\psi\|_\infty = \sup_{z\in\mathbb{D}} |\varphi(z)\psi(z)| < \sigma < 1.$$

Then we can find measurable disjoint subsets Ω_1 and Ω_2 in the unit disk such that $\int_{\Omega_1\cup\Omega_2} dA_\alpha(z) = 1$, and $|\varphi(z)| < \sqrt{\sigma}$, a.e. in Ω_1, and $|\psi(z)| < \sqrt{\sigma}$, a.e. in Ω_2. To see this we define

$$\Omega_1 = \{z : |\varphi(z)| < \sqrt{\sigma},\ a.e.\},$$

and

$$\Omega_2 = \{z : |\varphi(z)| \geq \sqrt{\sigma},\ a.e.\}.$$

Clearly if $z \notin \Omega_1$, then $z \in \Omega_2$ and $|\varphi(z)| \geq \sqrt{\sigma}$. Hence we must have $|\psi(z)| < \sqrt{\sigma}$ since otherwise we have $|\varphi(z)\psi(z)| \geq \sigma$ which is not possible. This argument will be used in the proof of the main result.

3. Main result

We begin by proving that if $\Phi = (\varphi, \psi)$ is a holomorphic mapping from the unit disk to $\mathbb{D}^2$, then the composition operator $C_\Phi : A^2_\alpha(\mathbb{D}^2) \to A^2_\alpha(\mathbb{D})$ is bounded.

Theorem 3.1. *Let $\Phi = (\varphi, \psi)$ where φ and ψ are analytic self-mappings of the unit disk satisfying $\|\varphi\psi\|_\infty < 1$. Then $C_\Phi : A^2_\alpha(\mathbb{D}^2) \to A^2_\alpha(\mathbb{D})$ is bounded.*

Sketch of proof. It is clear that

$$\|C_\Phi F\|^2_{A^2_\alpha(\mathbb{D})} = \int_{\Omega_1} |C_\Phi F|^2 + \int_{\Omega_2} |C_\Phi F|^2.$$

Then we approximate

$$\int_{\Omega_1} |C_\Phi F|^2$$

by σ, norm of C_ψ and norm of F in the Bergman space $A^2_\alpha(\mathbb{D}^2)$. Similarly, one approximates

$$\int_{\Omega_2} |C_\Phi F|^2$$

by σ, norm of C_φ, and norm of F in the Bergman space $A^2_\alpha(\mathbb{D}^2)$. Finally,

$$\|C_\Phi F\|^2_{A^2_\alpha(\mathbb{D})} \leq C_\sigma \Big(\|C_\varphi\|^2 + \|C_\psi\|^2\Big) \|F\|_{A^2_\alpha(\mathbb{D}^2)}.$$

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Oral Presentation

EXISTENCE AND CONVERGENCE OF FIXED POINT RESULTS FOR NONCYCLIC CONTRACTIONS IN REFLEXIVE BANACH SPACES

AKRAM SAFARI-HAFSHEJANI

Department of Pure Mathematics, Payame Noor University (PNU), P. O. Box: 19395-3697, Tehran, Iran
asafari@pnu.ac.ir

Abstract. In this paper, we study the existence of a fixed point for a noncyclic contraction map in a reflexive Banach space. The presented results extend and improve some recent results in the literature.

1. Introduction

Let A and B be nonempty subsets of a metric space (X, d). A self mapping $T : A \cup B \to A \cup B$ is said to be *noncyclic* provided that $T(A) \subseteq A$ and $T(B) \subseteq B$. We say that $(x, y) \in A \times B$ is an *optimal pair of fixed points* of the noncyclic mapping T provided that

$$Tx = x, \quad Ty = y \quad \text{and} \quad d(x, y) = d(A, B),$$

where $d(A, B) = \inf\{d(a, b) : \; a \in A, \; b \in B\}$.

In 2005, Anthony Eldred, Kirk and Veeremani [2] introduced noncyclic mappings and studied the existence of an optimal pair of fixed points of a given mapping.

In 2013, Abkar and Gabeleh [1] introduced noncyclic contraction mappings. As a result of theorem 2.7 of [6], for these mappings, the authors presented the following existence theorem.

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Theorem 1.1. *Let A and B be nonempty convex subsets of a uniformly convex Banach space X such that A is closed and let $T : A \cup B \to A \cup B$ be a noncyclic contraction map that is, there exists $c \in [0, 1)$ such that*

$$d(Tx, Ty) \leq cd(x, y) + (1 - c)d(A, B),$$

for all $x \in A$ and $y \in B$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique fixed point $x \in A$ such that $x_n \to x$.

In this paper, we study the existence of a fixed point for a noncyclic contraction map in a reflexive Banach space.

Here, we recall a definition and fact will be used in the next section.

Definition 1.2. [5] A Banach space X is said to be strictly convex if the following implication holds for all $x, y, p \in X$ and $R > 0$:

$$\left.\begin{array}{c} \|x - p\| \leq R \\ \|y - p\| \leq R \\ x \neq y \end{array}\right\} \Rightarrow \|\frac{x + y}{2} - p\| < R.$$

Theorem 1.3. [6] *Let A and B be nonempty closed subsets of a complete metric space (X, d). Let T be a noncyclic mapping on $A \cup B$ satisfying*

$$d(Tx, Ty) \leq cd(x, y),$$

for each $x \in A$ and $y \in B$ where $c \in [0, 1)$. Then T has a unique fixed point x in $A \cap B$ and the Picard iteration $\{T^n x_0\}$ converges to x for any starting point $x_0 \in A \cup B$.

2. Main results

The following results will be needed to prove the main theorems of this section.

Lemma 2.1. *Let A and B be nonempty subsets of the metric space (X, d) and let $T : A \cup B \to A \cup B$ be a noncyclic contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ and for $y_0 \in B$, define $y_{n+1} := Ty_n$ for each $n \geq 0$. Then $d(x_n, y_n) \to d(A, B)$ as $n \to \infty$.*

The next two results show the existence of a fixed point for a noncyclic contraction map in a reflexive Banach space.

Theorem 2.2. *Let A and B be nonempty weakly closed subsets of a reflexive Banach space X and let $T : A \cup B \to A \cup B$ be a noncyclic contraction map. Then there exists $(x, y) \in A \times B$ such that $\|x - y\| = d(A, B)$.*

Proof. If $d(A, B) = 0$, the result follows from Theorem 1.3. So, we assume that $d(A, B) > 0$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ and for $y_0 \in A$, define $y_{n+1} := Ty_n$ for each $n \geq 0$. By Lemma 2.2 of [6], the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. As X is reflexive and A is weakly closed, the sequence $\{x_n\}$ has a subsequence$\{x_{n_k}\}$ with $x_{n_k} \xrightarrow{w} x \in A$. As $\{y_{n_k}\}$ is bounded and B is weakly closed, we can say, without loss of generality, that $y_{n_k} \xrightarrow{w} y \in B$

as $k \to \infty$. Since $x_{n_k} - y_{n_k} \xrightarrow{w} x - y \neq 0$ as $k \to \infty$, there exists a bounded linear functional $f : X \to [0, +\infty)$ such that

$$\|f\| = 1 \quad and \quad f(x-y) = \|x-y\|.$$

For each $k \geq 1$, we have

$$|f(x_{n_k} - y_{n_k})| \leq \|f\|\|x_{n_k} - y_{n_k}\| = \|x_{n_k} - y_{n_k}\|.$$

Since

$$\lim_{k\to\infty} f(x_{n_k} - y_{n_k}) = f(x-y) = \|x-y\|,$$

it follows from Lemma 2.1 that

$$\|x-y\| = \lim_{k\to\infty} |f(x_{n_k} - y_{n_k})| \leq \lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = d(A,B).$$

Thus $\|x-y\| = d(A,B)$. □

Definition 2.3. [4] A mapping $F : C \subseteq X \to X$ is called demiclosed at y if, whenever $x_n \xrightarrow{w} x \in C$ and $Fx_n \xrightarrow{s} y \in X$, it follows that $Fx = y$.

Let I is the identity map, $I - T : A \cup B \to X$ is demiclosed at 0 if whenever x_n is a sequence in $A \cup B$ such that $x_{n_k} \xrightarrow{w} x \in A \cup B$ and $(I-T)x_n \xrightarrow{s} 0$ as $n \to \infty$, then $(I-T)x = 0$.

Theorem 2.4. *Let A and B be nonempty subsets of a reflexive Banach space X such that A is weakly closed and let $T : A \cup B \to A \cup B$ be a noncyclic contraction map. Then there exists $x \in A$ such that $Tx = x$ provided one of the following conditions is satisfied:*

(a) *T is weakly continuous on A;*
(b) *$I - T : A \cup B \to X$ is demiclosed at 0.*

Proof. If $d(A,B) = 0$, the result follows from Theorem 1.3. So, we assume that $d(A,B) > 0$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. By Lemma 2.2 of [6], the sequence $\{x_n\}$ is bounded. As X is reflexive and A is weakly closed, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ with $x_{n_k} \xrightarrow{w} x \in A$ as $k \to \infty$.

(a) Since T is weakly continuous on A and $T(A) \subseteq A$, we have $x_{n_k+1} \xrightarrow{w} Tx \in A$ as $k \to \infty$. Thus $x_{n_k} - x_{n_k+1} \xrightarrow{w} x - Tx$ as $k \to \infty$. We assume the contrary, $x - Tx \neq 0$. Since $x_{n_k} - x_{n_k+1} \xrightarrow{w} x - Tx \neq 0$ as $k \to \infty$, there exists a bounded linear functional $f : X \to [0, +\infty)$ such that

$$\|f\| = 1 \quad and \quad f(x - Tx) = \|x - Tx\|.$$

For each $k \geq 1$, we have

$$|f(x_{n_k} - x_{n_k+1})| \leq \|f\|\|x_{n_k} - x_{n_k+1}\| = \|x_{n_k} - x_{n_k+1}\|.$$

Since

$$\lim_{k\to\infty} f(x_{n_k} - x_{n_k+1}) = f(x - Tx) = \|x - Tx\|,$$

it follows from Lemma 2.1 that

$$\|x - Tx\| = \lim_{k\to\infty} |f(x_{n_k} - x_{n_k+1})| \leq \lim_{k\to\infty} \|x_{n_k} - x_{n_k+1}\| = 0.$$

Thus $\|x - Tx\| = 0$, a contradiction.

(b) By Lemma 2.1, we have

$$\|x_{n_k} - Tx_{n_k}\| = \|x_{n_k} - x_{n_k+1}\| \to 0$$

as $k \to \infty$. So $(I - T)x_{n_k} \xrightarrow{s} 0$ as $k \to \infty$. As $I - T : A \cup B \to X$ is demiclosed at 0, it follows that $(I - T)x = 0$. Hence $Tx = x$. □

The next result show the existence and uniqueness of a best proximity point for a cyclic contraction map in a reflexive and strictly Banach space. This theorem guarantees the uniqueness in Theorem 3.5 of [3].

Theorem 2.5. *Let A and B be nonempty closed and convex subsets of a reflexive and strictly convex Banach space X and let $T : A \cup B \to A \cup B$ be a noncyclic contraction map. If $(A - A) \cap (B - B) = \{0\}$, then there exists a unique optimal pair of fixed points $(x, y) \in A \times B$ for T.*

Proof. If $d(A, B) = 0$, the result follows from Theorem 1.3. So, we assume that $d(A, B) > 0$. Since A is closed and convex, it is weakly closed. It follows from Theorem 2.2 that there exists $(x, y) \in A \times B$ such that $\|x - y\| = d(A, B)$. To show the uniqueness of (x, y), suppose that there exists another $(x', y') \in A \times B$ such that $\|x' - y'\| = d(A, B)$. As $(A - A) \cap (B - B) = \{0\}$ we conclude that $x - x' \neq y - y'$ and so $x - y \neq x' - y'$. Since A and B are both convex, it follows from the strict convexity of X that

$$\|\frac{x + x'}{2} - \frac{y + y'}{2}\| = \|\frac{x - y + x' - y'}{2} - 0\| < d(A, B),$$

a contradiction. As

$$\|Tx - Ty\| = \|x - y\| = d(A, B),$$

we conclude, from the uniqueness of (x, y), that $(Tx, Ty) = (x, y)$. Thus $Tx = x$ and $Ty = y$. □

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Oral Presentation

EXISTENCE AND CONVERGENCE OF BEST PROXIMITY POINT RESULTS FOR CYCLIC QUASI-CONTRACTIONS IN REFLEXIVE BANACH SPACES

AKRAM SAFARI-HAFSHEJANI

Department of Pure Mathematics, Payame Noor University (PNU), P. O. Box: 19395-3697, Tehran, Iran
asafari@pnu.ac.ir

Abstract. In this paper, we study the existence of a best proximity point for a cyclic quasi-contraction map in a reflexive Banach space. The presented results extend and improve some recent results in the literature.

1. Introduction

In 2009 Al-Thagafi and Shahzad [1] prove the existence of a best proximity point for a cyclic contraction map in a reflexive Banach space.

Theorem 1.1. [1, Theorem 9] *Let A and B be nonempty weakly closed subsets of a reflexive Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Then there exists $(x, y) \in A \times B$ such that $\|x - y\| = d(A, B)$.*

Definition 1.2. [4] Let A and B be nonempty subsets of a normed space X and T be a cyclic map on $A \cup B$. We say that T satisfies the proximal property if

$$x_n \xrightarrow{w} x \in A \cup B \quad \textit{and} \quad \|x_{n_k} - Tx_{n_k}\| \to d(A, B) \Longrightarrow \|x - Tx\| = d(A, B).$$

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Theorem 1.3. [1, Theorem 10] *Let A and B be nonempty subsets of a reflexive Banach space X such that A is weakly closed and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Then there exists $x \in A$ such that $\|x - Tx\| = d(A, B)$ provided one of the following conditions is satisfied:*

(a) *T is weakly continuous on A;*
(b) *T satisfies the proximal property.*

Theorem 1.4. [1, Theorem 12] *Let A and B be nonempty subsets of a reflexive and strictly convex Banach space X such that A is closed and convex and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Then there exists a unique $x \in A$ such that $T^2x = x$ and $\|x - Tx\| = d(A, B)$ provided one of the following conditions is satisfied:*

(a) *T is weakly continuous on A;*
(b) *T satisfies the proximal property.*

Theorem 1.5. [6] *Let A and B be nonempty and closed subsets of a complete metric space (X, d). Let T be a cyclic mapping on $A \cup B$ such that*

$$d(Tx, Ty) \leq c \max\{d(x, y), d(x, Tx), d(y, Ty)\},$$

for all $x \in A$ and $y \in B$ where $c \in [0, 1)$. Then T has a unique fixed point x in $A \cap B$ and the Picard iteration $\{T^n x_0\}$ converges to x for any starting point $x_0 \in A \cup B$.

2. MAIN RESULTS

The following results will be needed to prove the main theorems of this section.

Lemma 2.1. *Let A and B be nonempty subsets of the metric space (X, d) and let $T : A \cup B \to A \cup B$ be a cyclic quasi-contraction map, that is there exists $\lambda \in [0, 1)$ such that*

$$d(Ty, Tx) \leq \lambda max\{d(x, y), d(x, Tx), d(Ty, y)\} + (1 - \lambda)d(A, B)$$

for all $x \in A$ and $y \in B$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then $d(x_{2n}, x_{2n+1}) \to d(A, B)$ as $n \to \infty$.

The next two results that are extentions of Theorems 1.1 and 1.3, show the existence of a best proximity point for a cyclic qusi-contraction map in a reflexive Banach space.

Theorem 2.2. *Let A and B be nonempty weakly closed subsets of a reflexive Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic quasi-contraction map. Then there exists $(x, y) \in A \times B$ such that $\|x - y\| = d(A, B)$.*

Proof. If $d(A, B) = 0$, the result follows from Theorem 1.5. So, we assume that $d(A, B) > 0$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ and for each $n \geq 0$. By Lemma 3.2 of [3], the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded. As X is reflexive and A is weakly closed, the sequence $\{x_{2n}\}$ has a subsequence$\{x_{2n_k}\}$ with $x_{2n_k} \xrightarrow{w} x \in A$. As $\{x_{2n_k+1}\}$ is bounded and B is weakly closed, we can say, without loss of generality, that $x_{2n_k+1} \xrightarrow{w} y \in B$ as $k \to \infty$. Since $x_{2n_k} - x_{2n_k+1} \xrightarrow{w} x - y \neq 0$ as $k \to \infty$, there exists a bounded linear functional $f : X \to [0, +\infty)$ such that

$$\|f\| = 1 \quad and \quad f(x - y) = \|x - y\|.$$

For each $k \geq 1$, we have

$$|f(x_{2n_k} - x_{2n_k+1})| \leq \|f\| \|x_{2n_k} - x_{2n_k+1}\| = \|x_{2n_k} - x_{2n_k+1}\|.$$

Since

$$\lim_{k\to\infty} f(x_{2n_k} - x_{2n_k+1}) = f(x - y) = \|x - y\|,$$

it follows from Lemma 2.1 that

$$\|x - y\| = \lim_{k\to\infty} |f(x_{2n_k} - x_{2n_k+1})| \leq \lim_{k\to\infty} \|x_{2n_k} - x_{2n_k+1}\| = d(A, B).$$

Thus $\|x - y\| = d(A, B)$. □

The following theorem is proved in a completely similar way to the proof of theorem 1.3.

Theorem 2.3. *Let A and B be nonempty subsets of a reflexive Banach space X such that A is weakly closed and let $T : A \cup B \to A \cup B$ be a cyclic quasi-contraction map. Then there exists $x \in A$ such that $\|x - Tx\| = d(A, B)$ provided one of the following conditions is satisfied:*

(a) *T is weakly continuous on A.*
(b) *T satisfies the proximal property.*

Proof. If $d(A, B) = 0$, the result follows from Theorem 1.5. So, we assume that $d(A, B) > 0$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. By Lemma 3.2 of [3], the sequence $\{x_{2n}\}$ is bounded. As X is reflexive and A is weakly closed, the sequence $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ with $x_{2n_k} \xrightarrow{w} x \in A$ as $k \to \infty$.

(a) Since T is weakly continuous on A and $T(A) \subseteq B$, we have $x_{2n_k+1} \xrightarrow{w} Tx \in B$ as $k \to \infty$. Thus $x_{2n_k} - x_{2n_k+1} \xrightarrow{w} x - Tx$ as $k \to \infty$. Since $x_{2n_k} - x_{2n_k+1} \xrightarrow{w} x - Tx \neq 0$ as $k \to \infty$, there exists a bounded linear functional $f : X \to [0, +\infty)$ such that

$$\|f\| = 1 \quad and \quad f(x - Tx) = \|x - Tx\|.$$

For each $k \geq 1$, we have

$$|f(x_{n_k} - x_{n_k+1})| \leq \|f\| \|x_{2n_k} - x_{2n_k+1}\| = \|x_{2n_k} - x_{2n_k+1}\|.$$

Since

$$\lim_{k\to\infty} f(x_{2n_k} - x_{2n_k+1}) = f(x - Tx) = \|x - Tx\|,$$

it follows from Lemma 2.1 that

$$\|x - Tx\| = \lim_{k\to\infty} |f(x_{2n_k} - x_{2n_k+1})| \leq \lim_{k\to\infty} \|x_{2n_k} - x_{2n_k+1}\| = d(A, B).$$

Thus $\|x - Tx\| = d(A, B)$.

(b) By Lemma 2.1 , we have

$$\|x_{2n_k} - Tx_{2n_k}\| = \|x_{2n_k} - x_{2n_k+1}\| \to d(A,B).$$

as $k \to \infty$. As T satisfies the proximal property, it follows that $\|x - Tx\| = d(A,B)$. ☐

The next result that is extention of Theorem 1.4,shows the existence and uniqueness of a best proximity point for a cyclic quasi contraction map in a reflexive and strictly Banach space.

Theorem 2.4. *Let A and B be nonempty subsets of a reflexive and strictly convex Banach space X such that A is closed and convex let $T : A \cup B \to A \cup B$ be a cyclic quasi-contraction map. Then there exists a unique $x \in A$ such that $T^2x = x$ and $\|x - Tx\| = d(A,B)$ provided one of the following conditions is satisfied:*

(a) *T is weakly continuous on A.*
(b) *T satisfies the proximal property.*

Proof. If $d(A,B) = 0$, the result follows from Theorem 1.5. So, we assume that $d(A,B) > 0$. Since A is closed and convex, it is weakly closed. It follows from Theorem 2.3 that there exists $x \in A$ such that $\|x - Tx\| = d(A,B)$. Also

$$\begin{aligned}\|T^2x - Tx\| &\leq \lambda \max\left\{\|Tx - x\|, \|T^2x - Tx\|\right\} + (1-\lambda)d(A,B)\\ &= \lambda\|T^2x - Tx\| + (1-\lambda)d(A,B),\end{aligned}$$

and so $\|T^2x - Tx\| = d(A,B)$. In fact, $T^2x = x$. To see this, assume that $T^2x \neq x$. It follows, from the convexity of A and the strict convexity of X, that

$$\|\frac{T^2x + x}{2} - Tx\| < d(A,B),$$

a contradiction. A similar argument shows the uniqueness of x follows as ☐

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SOME MINIMAX THEOREMS FOR VECTOR-VALUED FUNCTIONS IN G-CONVEX SPACES

MARYAM SALEHNEJAD* AND MAHDI AZHINI

Department of Mathematics, Firuzkuh branch, Islamic Azad University, Tehran, Iran.
Salehnejad_m@yahoo.com
Department of Mathematics, Science and Research Branc, Islamic Azad University, Tehran, Iran.
mahdi.azhini@gmail.com

Abstract. In this article, we give some generalized minimax inequalities for vector-valued functions by means of the generalized KKM theorem.
Keywords: G-convex space, Generalized KKM Map, Cone-γ-generalized quasi-convex (concave), Minimax Inequality.

1. INTRODUCTION

The minimax inequality plays a significant role in many fields, such as variational inequalities, game theory, mathematical economics, optimization theory and fixed point theory. Because of widespread use, this inequality has been extended in variety of ways. (For example, see M. Salehnejad and M. Azhini [8], Ding and Tan [3], Horvath [7], Georgiev and Tanaka [6])
At the beginning, the consideration of minimax theorems were mainly devoted to the study of real and vector-valued functions in topological vector spaces. Motivated by the well-known works of Horvath [7], there here appeared many generalizations of the concept of convex subset of a tapological vector space. The most general one seems to be that of the generalized

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* Speaker.

convex space or G-convex space introduced by Park and Kim [10, 11] which extends many generalizd convex structures on topological vector space. This will be the framework in which we obtain in this work some minimax inequalities for vector-valued functions. (See Chen [2], Chang et al [1], M.G.Yang et al [14])

2. Preliminaries

Let X be a topological space and E be a nonempty subset of X. We denote by $\langle E\rangle$, the family of all nonempty finite subsets of E. Let Δ_n be the standard n–simplex $(e_1, \ldots, e_n)$ in R^{n+1}. If J is a nonempty subset of $\{0, 1, \ldots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j, j \in J\}$. The following notion of a generalized convex (or G–convex) space was introduced by Park and Kim [12]. Let X be a topological space and D is a nonempty set, $(X, D; \Gamma)$ is said to be a G–convex space if for each $A = \{a_0, \ldots, a_n\} \in \langle D\rangle$, there exists a subset $\Gamma(A) = \Gamma_A$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \subset A$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.
When $D \subset X$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$ and if $X = D$, we write $(X; \Gamma)$ in place of $(E, E; \Gamma)$. For a G–convex space $(X \supset D; \Gamma)$,

(1) a subset Y of X is said to be Γ–convex if for each $N \in \langle D\rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$;
(2) the Γ–convex hull of a subset Y of X, denoted by $\Gamma - Co(Y)$, is defined by

$$\Gamma - Co(Y) = \bigcap \{Z \subset X : Z \text{ is a } \Gamma\text{–convex subset containing } Y\}.$$

Definition 2.1. If V is a real vector space, a nonempty subset $P \subset V$ is a cone if for every $x \in P$ and for every $\lambda \geq 0$, we have $\lambda x \in P$. The cone P is called

(1) convex if for all $x_1, x_2 \in P$, $x_1 + x_2 \in P$
(2) pointed if $P \cap (-P) = \{0\}$
(3) proper if $P \neq \{0\}$ and $P \neq V$
(4) solid if int$P \neq \emptyset$ (where int$P$ denotes the interior of the set P).

If P is a convex cone of a real vector space V, the relation $\preceq_p$ define below is a (partial) vector ordering of V:

$$x \preceq_p y \;\Leftrightarrow\; y - x \in P,\ \forall x, y \in V$$

Definition 2.2. Let Y be a nonempty set and E be a nonempty subset of a G–convex space $(X, D; \Gamma)$. $T : Y \to 2^E$ is called a generalized KKM mapping if for any finite set $\{y_0, y_1, \ldots, y_n\} \subset Y$, there exists $\{x_0, x_1, \ldots, x_n\} \in \langle E \cap D\rangle$ such that for any subset $\{x_{i_0}, x_{i_1}, \ldots, x_{i_k}\} \subset \{x_0, \ldots, x_n\}$, $0 \leq k \leq n$, we have

$$\Gamma(\{x_{i_0}, \ldots, x_{i_k}\}) \subset \bigcup_{j=0}^{k} T(y_{i_j})$$

Theorem 2.3 ([8])**.** *Let E be a nonempty Γ–convex subset of a G–convex space $(X;\Gamma)$ and $G : E \to 2^X$ be such that for any $y \in E$, $G(y)$ is compactly closed. Then:*

(1) *If G is a generalized KKM mapping, then the family of sets $\{G(y) : y \in E\}$ has the finite intersection property.*
(2) *If the family $\{G(y) : y \in E\}$ has the finite intersection property and $\Gamma(x) = \{x\}$ for each $x \in X$, then G is a generalized KKM mapping.*

Definition 2.4. Let Y be a nonempty set and $(X;\Gamma)$ be a G-convex space and E, F are nonempty subsets of X, Y, respectively. The bi-function $\varphi : E\times F \to V$ is said to be cone-γ-generalized quasi-convex (concave) in second component for some $\gamma \in V$, if for any finite subset $\{y_0, y_1, \dots, y_n\}\subset F$, there exists a finite subset $\{x_0, x_1, \dots, x_n\} \subset E$ such that for any subset $\{x_{i_0}, ..., x_{i_k}\} \subset \{x_0, x_1, \dots, x_n\}$ and any $x^* \in co_{\mathcal{C}}\{x_{i_0}, ..., x_{i_k}\}$, there exists $j \in \{0, ..., k\}$ such that

$$\varphi(x^*, y_{i_j}) \in \gamma + P$$
$$(\ \varphi(x^*, y_{i_j}) \in \gamma - P\ ).$$

3. Main Results

In this section, we present Minimax inequalities in G-convex spaces for vector-valued functions. In sequel, suppose that X is a Hausdorff topological space, $(X;\Gamma)$ is a G–convex space and E, F are nonempty Γ–convex subsets of X. Also, (V, P) is an ordered topological vector space and P is a closed pointed convex cone such that $P \neq \phi$.

Definition 3.1. A function $\varphi : X \longrightarrow V$ is called lower [resp. upper] semi-continuous if for every $\gamma \in V$, the set $\{x \in X : \ f(x) \in \gamma - P\}$ [resp. $\{x \in X : \ f(x) \in \gamma + P\}$] is closed in X. [9]

Theorem 3.2. *Let φ and ψ be two functions from $E \times F$ to V such that:*

(1) *$\varphi(x, y)$ is lower semi-continuous in x, for each $y \in F$;*
(2) *$\psi(x, y)$ is cone$-\gamma-$generalized quasi-concave in y, for some $\gamma \in V$;*
(3) *$\varphi(x, y) \preceq \psi(x, y)$ for all $(x, y) \in E \times F$;*
(4) *for some $y_0 \in F$, $\{x \in E \ : \ \varphi(x, y_0) \in \gamma - P\}$ is a compact subset of E.*

Then there exists an $\bar{x} \in E$ such that

$$\varphi(\bar{x}, y) \in \gamma - P \text{ for all } y \in F.$$

Remark 3.3. Fan's Minimax inequality can be deduced from the above Theorem if $X = Y$, $E = F$, $V = \mathbb{R}$ and $\varphi = \psi$. (See [4])

Theorem 3.4. *Let $\varphi : E \times E \to V$ and $\gamma \in V$ be such that*

(1) *for each $x \in E$, $\varphi(x, y)$ is a lower semi-continuous function of y on each non-empty compact subset C of E;*
(2) *$\varphi(x, y)$ is cone$-\gamma-$generalized quasi-concave in x;*

(3) *there exist a non-empty compact convex subset M of E and a non-empty compact K of E such that for each $y \in E \setminus K$, there is an $x \in M$ with $\varphi(x, y) \notin \gamma - P$.*

Then there exists $\hat{y} \in K$ such that $\varphi(x, \hat{y}) \in \gamma - P$ for all $x \in E$.

As an immediate result of Theorem 3.4, we can conclude the following minimax inequality theorem, which in turn generalizes minimax inequality due to Ding and Tan [3] and Fan [5], to vector-valued and cone$-\gamma-$generalized quasi-concave functions.

Theorem 3.5. *Let $\gamma \in V$ and $\varphi, \psi : E \times E \to V$ be two functions such that*

(1) *$\varphi(x, y) \preceq \psi(x, y)$ for all $(x, y) \in E \times E$*
(2) *for each fixed $x \in E$, $\varphi(x, y)$ is a lower semi-continuous function of y on each non-empty compact subset C of E;*
(3) *$\psi(x, y)$ is cone$-\gamma-$generalized quasi-concave in x;*
(4) *there exist a non-empty compact convex subset M of E and a non-empty compact subset K of E such that for each $y \in E \setminus K$, there is an $x \in M$ with $\varphi(x, y) \notin \gamma - P$.*

Then there exists $\hat{y} \in K$ such that $\varphi(x, \hat{y}) \in \gamma - P$ for all $x \in E$.

The following is a generalization of minimix inequality due to Tan and Yuan [13], to vector-valued and cone$-\gamma-$generalized quasi-concave functions.

Theorem 3.6. *Let $\gamma \in V$ and $\varphi, \psi : E \times F \to V$ be two functions such that*

(1) *$\varphi(x, y) \preceq \psi(x, y)$ for all $(x, y) \in E \times F$*
(2) *for each fixed $y \in F$, $\varphi(x, y)$ is a lower semi-continuous function of x on each non-empty compact subset C of E;*
(3) *$\psi(x, y)$ is cone$-\gamma-$generalized quasi-concave in y;*
(4) *there exists a non-empty compact subset K of E and $y^* \in F$ such that $\psi(x, y^*) \notin \gamma - P$ for each $x \in E \setminus K$.*

Then there exists an $\hat{x} \in K$ such that $\varphi(\hat{x}, y) \in \gamma - P$ for all $y \in F$.

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POWER BOUNDED AND MEAN ERGODIC OPERATORS ON BLOCH TYPE AND ZYGMUND TYPE SPACES

AMIR H. SANATPOUR* AND ZAHRA SADAT HOSSEINI

Department of Mathematics, Kharazmi University, Tehran 15618-36314, Iran
a_sanatpour@khu.ac.ir; sadathosseini213@gmail.com

Abstract. In this paper we investigate power bounded and mean ergodic properties of composition operators on Bloch type spaces and also on Zygmund type spaces. We study their relation with a corresponding composition operator on analytic Lipschitz algebras and also on differentiable Lipschitz algebras, respectively.

1. Introduction

Let X be a Banach space and $B(X)$ denote the space of all bounded operators on X. An operator T on X is called *power bounded* if the sequence $\{T^n\}_{n\in\mathbb{N}}$ is bounded in $B(X)$. The operator T on X is called *mean ergodic* if for each $x \in X$ the sequence $\{T_{[n]}(x)\}_{n\in\mathbb{N}}$ is convergent to some $Lx \in X$ for some $L \in B(X)$, where

$$T_{[n]} := \frac{1}{n}\sum_{k=1}^{n} T^k, \qquad \text{for each } n \in \mathbb{N}.$$

Theory of mean ergodic operators on Banach spaces are related to the theory of bases in Banach spaces. It is known that a Banach space X with a basis is reflexive if and only if every power bounded operator on X is mean ergodic. This topic was first studied by Bonet and Domanski in 2011 on $H(U)$ for a Stein manifold U [1]. Later, many authors investigated power bounded

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$*$ Sspeaker.

and mean ergodic properties of different type of operators between various type of Banach spaces. In this paper we mainly focus on certain type of operators called composition operators, defined as follows.

Let $\mathbb{D}$ denote the open unit disc of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space all complex-valued analytic functions on $\mathbb{D}$. For an analytic selfmap φ of $\mathbb{D}$, the composition operator induced by φ, denoted by C_φ, is defined by

$$C_\varphi(f) = f \circ \varphi, \qquad \text{for each } f \in H(\mathbb{D}).$$

Composition operators appear in the study of dynamical systems and also in characterizing isometries on many analytic function spaces. Composition operators between various spaces of analytic functions have been studied by many authors, see for example [2, 3, 4] and the references therein.

Power bounded and mean ergodic composition operators on different Banach spaces have been studied by many authors, see for example [1, 5] and the references therein. In the next chapter, we investigate power bounded and mean ergodic properties of composition operators on Bloch type spaces and also on Zygmund type spaces. Indeed, we study their relation with a corresponding composition operator on analytic Lipschitz algebras and also on differentiable Lipschitz algebras, respectively.

2. Main result

For each $0 < \alpha < \infty$, the Bloch type space of order α, denoted by $\mathcal{B}^\alpha$, is the space of all functions $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1 - |z|)^\alpha |f'(z)| < \infty.$$

The Bloch type space $\mathcal{B}^\alpha$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|)^\alpha |f'(z)|, \quad (f \in \mathcal{B}^\alpha).$$

Let $A(\overline{\mathbb{D}})$ denote the classic disc algebra containing of all continuous functions on $\overline{\mathbb{D}}$ which are analytic on $\mathbb{D}$. For each $0 < \alpha \leq 1$, the analytic Lipschitz algebra of order α, denoted by $Lip_A(\overline{\mathbb{D}}, \alpha)$, is algebra of all functions $f \in A(\overline{\mathbb{D}})$ with

$$\rho_\alpha(f) = \sup_{\substack{z,w\in\overline{\mathbb{D}} \\ z\neq w}} \frac{|f(z) - f(w)|}{|z - w|^\alpha} < \infty.$$

The analytic Lipschitz algebra $Lip_A(\overline{\mathbb{D}}, \alpha)$ is a Banach algebra with the norm

$$\|f\|_{Lip^\alpha} = \|f\|_\infty + \rho_\alpha(f), \quad (f \in Lip_A(\overline{\mathbb{D}}, \alpha)),$$

where $\|f\|_\infty = \sup_{z\in\overline{\mathbb{D}}} |f(z)|$.

It is known that for each $0 < \alpha < 1$, every $f \in \mathcal{B}^\alpha$ has a unique continuous extension to some $F \in Lip_A(\overline{\mathbb{D}}, 1 - \alpha)$, see [3]. We next give the relation between power boundedness of a composition operator on Bloch type spaces and power boundedness of a corresponding composition operator on analytic Lipschitz algebras.

Theorem 2.1. *Let $0 < \alpha < 1$ and φ be an analytic selfmap of $\mathbb{D}$. Then, the composition operator $C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ is power bounded if and only if the composition operator $C_\phi : Lip_A(\overline{\mathbb{D}}, 1-\alpha) \to Lip_A(\overline{\mathbb{D}}, 1-\alpha)$ is power bounded, where ϕ is the unique extension of φ to $\overline{\mathbb{D}}$.*

We next give the result of Theorem 2.1 for the mean ergodicity of a composition operator between Bloch type spaces.

Theorem 2.2. *Let $0 < \alpha < 1$ and φ be an analytic selfmap of $\mathbb{D}$. Then, the composition operator $C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ is mean ergodic if and only if the composition operator $C_\phi : Lip_A(\overline{\mathbb{D}}, 1-\alpha) \to Lip_A(\overline{\mathbb{D}}, 1-\alpha)$ is mean ergodic, where ϕ is the unique extension of φ to $\overline{\mathbb{D}}$.*

Let $0 < \alpha < \infty$. The Zygmund type space of order α, denoted by $\mathcal{Z}^\alpha$, consists of those analytic functions f on $\mathbb{D}$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|)^\alpha|f''(z)| < \infty.$$

The Zygmund type space $\mathcal{Z}^\alpha$ is a Banach space, with the norm

$$\|f\|_{\mathcal{Z}^\alpha} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|)^\alpha|f''(z)|, \quad (f \in \mathcal{Z}^\alpha).$$

More generally, for each $n \in \mathbb{N}$ and $0 < \alpha < \infty$, the space of all functions $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|)^\alpha|f^{(n+1)}(z)| < \infty,$$

is called n-Zygmund type space and is denoted by $\mathcal{Z}_n^\alpha$, see [3]. The space $\mathcal{Z}_n^\alpha$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{Z}_n^\alpha} := |f(0)|+|f'(0)|+\cdots+|f^{(n)}(0)|+\sup_{z\in\mathbb{D}}(1-|z|)^\alpha|f^{(n+1)}(z)|, \quad (f \in \mathcal{Z}_n^\alpha).$$

For each $n \in \mathbb{N}$, differentiable Lipschitz algebra of order n, denoted by $Lip^n(X, \alpha)$, is the algebra of all complex-valued functions f on $\overline{\mathbb{D}}$ whose derivatives up to order n exist and $f^{(k)} \in Lip(\overline{\mathbb{D}}, \alpha)$ for each $0 \leq k \leq n$. The algebra $Lip^n(\overline{\mathbb{D}}, \alpha)$ is a Banach algebra equipped with the norm

$$\|f\|_{Lip^n(\overline{\mathbb{D}},\alpha)} := \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\infty + \rho_\alpha(f^{(k)})}{k!}, \quad (f \in Lip^n(\overline{\mathbb{D}}, \alpha)).$$

As in the case of Bloch type spaces, it is known that for each $n \in \mathbb{N}$ and $0 < \alpha < 1$, every $f \in \mathcal{Z}_n^\alpha$ has a unique continuous extension to some $F \in Lip^n(\overline{\mathbb{D}}, \alpha)$, see [3]. Our next result gives the relation between power boundedness of a composition operator on n-Zygmund type spaces and power boundedness of a corresponding composition operator on differentiable Lipschitz algebras of order n.

Theorem 2.3. *Let $n \in \mathbb{N}$, $0 < \alpha < 1$ and φ be an analytic selfmap of $\mathbb{D}$. Then, the composition operator $C_\varphi : \mathcal{Z}_n^\alpha \to \mathcal{Z}_n^\alpha$ is power bounded if and only* *if the composition operator* $C_\phi : Lip^n(\overline{\mathbb{D}}, 1-\alpha) \to Lip^n(\overline{\mathbb{D}}, 1-\alpha)$ *is power bounded, where* ϕ *is the unique extension of* φ *to* $\overline{\mathbb{D}}$.

As the final result, we next consider mean ergodicity of a composition operator between n-Zygmund type spaces.

Theorem 2.4. *Let* $n \in \mathbb{N}$, $0 < \alpha < 1$ *and* φ *be an analytic selfmap of* $\mathbb{D}$. *Then, the composition operator* $C_\varphi : \mathcal{Z}_n^\alpha \to \mathcal{Z}_n^\alpha$ *is mean ergodic if and only if the composition operator* $C_\phi : Lip^n(\overline{\mathbb{D}}, 1-\alpha) \to Lip^n(\overline{\mathbb{D}}, 1-\alpha)$ *is mean ergodic, where* ϕ *is the unique extension of* φ *to* $\overline{\mathbb{D}}$.

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Oral Presentation **$*$: Speaker**

CONVOLUTION, ERROR FUNCTION AND A NEW SPECIAL FUNCTION IN CLASS OF UNIVALENT ANALYTIC FUNCTION

SEYED HADI SAYEDAIN BOROUJENI* AND SHAHRAM NAJAFZADEH

Department of Mathematics, Payame Noor University, Tehran, Iran.
h.sayedain@pnu.ac.ir, hadisayedain@gmail.com
shnajafzadeh44@pnu.ac.ir, najafzadeh1234@yahoo.ie

Abstract. The main objective of this paper is to introduce a new special class of analytic univalent functions based on a combination of the Error function and a new function, that we create with the help of convolution. we examine several properties of this class, such as, Weighted mean, Coefficient estimate and extreme points.

1. Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Denote by $\mathcal{A}$ the well-known class of analytic and normalized functions of in $\mathbb{U}$. we note that each function f in $\mathcal{A}$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}, a_n \in \mathbb{C}). \tag{1.1}$$

we say that a function f is univalent in $\mathbb{U}$ if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in \mathbb{U}$ with $z_1 \neq z_2$. The family of all univalent functions f in $\mathbb{U}$ is denoted by $\mathcal{S}$ [2, 4]. The subclass of $\mathcal{A}$ create withe changing negative coefficients and are of the type

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$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0). \tag{1.2}$$

The convolution or Hadamard product $f(z)$ and $g(z)$ for f to form (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$ for more details see [2].
The Error Function and Subclasses of Analytic Univalent Functions introduce by Sayedain and Najafzadeh [5], is form

$$\begin{aligned} \mathrm{E_r f(z)} &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} z^{2n+1} \\ &= z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} z^n, \quad (z \in \mathbb{C}). \end{aligned} \tag{1.3}$$

Let $h(z) = z + (\frac{3}{e} - 2)z^n$, $(n = 0, 1, 2, ...)$ and the Taylor series of this h

$$h(z) = z - \sum_{n=2}^{\infty} \frac{(-1)^n (2n-1)}{n!} z^n, \tag{1.4}$$

Definition 1.1. The function $H(z)$ denote by convolution $h(z)$ and $\mathrm{E_r f(z)}$

$$\begin{aligned} H(z) &= h(z) * (2z - \mathrm{E_r(f)}) * \mathrm{f(z)} \\ &= z - \sum_{n=2}^{\infty} \frac{1}{n((n-1)!)^2} a_n z^n. \end{aligned} \tag{1.5}$$

Where $f(z)$ is (1.2). new, a function of the form (1.2) is in the class $\mathcal{W}_Q(a, b)$ if it satisfies the condition

$$\mathrm{Re}\left\{ \frac{H(z) + zH'(z) + az^2 H''(z) - z}{azH'(z) + (1-b)H(z)} \right\} > Q, \quad {\scriptstyle (0 \leq Q < 1)}. \tag{1.6}$$

Where $0 \leq a, b \leq 1$, $a < b$ and $H(z)$ are give by (1.5), also $H'(z), H''(z)$ are first and second order derivatives, respectively [3].

2. MAIN RESULTS

In the folloing theorem, we express a condition for the functions that belong to the class $\mathcal{W}_Q(a, b)$.

Theorem 2.1. *Let $f(z)$ of the form*(1.2)*. f belong to the class $\mathcal{W}_Q(a, b)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{\frac{n+1}{n} + a(n(Q+1)) + Q(1-b)}{((n-1)!)^2} a_n \leq 1 + Q(b - a - 1), \quad {\scriptstyle (0 \leq Q < 1)}. \tag{2.1}$$

Proof. By calculating the derivatives of the first and second order and placing in (1.5) and also consider "z" as a real number that is moved to $z \to 1^-$ we have

$$\frac{1 - \sum_{n=2}^{\infty} \frac{a(n-1) + \frac{1}{n} + 1}{((n-1)!)^2} a_n}{(1+a-b) - \sum_{n=2}^{\infty} \frac{an - b + 1}{((n-1)!)^2} a_n} > Q$$

$$Q(b-a-1) + 1 - \sum_{n=2}^{\infty} \frac{a(n(Q+1)) + Q(1-b) + \frac{1}{n} + 1}{((n-1)!)^2} a_n \geq 0$$

So, we proved that if $f(z)$ belong to the class $\mathcal{W}_Q(a,b)$ the related (2.1) holds. Conversely, it is easily proven. see [5]. $\square$

The results is sharp for example $g(z) = z - \dfrac{1 + Q(b-a-1)}{Q(2a-b+1) + 2a + 1.5} z^2$.

Theorem 2.2. *$\mathcal{W}_Q(a,b)$ is a convex set.*

Proof. It is enough to show for $f_i(z)$ belong to the class $\mathcal{W}_Q(a,b)$, then $\sum_{i=1}^{m} \lambda_i f_i(z) \in \mathcal{W}_Q(a,b)$ where $\sum_{i=1}^{m} \lambda_i = 1$ and $\lambda_i \geq 0$. $\square$

3. ON GEOMETRIC PROPERTIES

In the following theorem, we introduce the functions that belong to the class $\mathcal{W}_Q(a,b)$ and are the extreme points of the set. we will show that they have such a property [1].

Theorem 3.1. *Let $f_n(z) = z - \dfrac{((n-1)!)^2(1 + Q(b-a-1))}{a(n(Q+1)) + Q(1-b) + \frac{1}{n} + 1} z^n$, $(n = 2, 3, ...)$ also $f_1(z) = z$. then $\sum_{n=1}^{\infty} \mu_n f_n(z) \in \mathcal{W}_Q(a,b)$ if and only if $\sum_{n=1}^{\infty} \mu_n = 1$ and $\mu_n \geq 0$.*

Proof. Let assume first $\sum_{n=1}^{\infty} \mu_n f_n(z) \in \mathcal{W}_Q(a,b)$ for $\sum_{n=1}^{\infty} \mu_n = 1$ and $\mu_n \geq 0$. we will show that $f_n \in \mathcal{W}_Q(a,b)$ $(n = 2, 3, ...)$. Refer theorem(2.1)

$$a_n \leq \frac{1 + Q(b-a-1)((n-1)!)^2}{a(n(Q+1)) + Q(1-b) + \frac{1}{n} + 1}$$

therefore by letting

$$\mu_n = \frac{a(n(Q+1)) + Q(1-b) + \frac{1}{n} + 1}{1 + Q(b-a-1)((n-1)!)^2} a_n$$

and that $\mu_1 = 1-(\mu_2+\mu_3+...)$ we conclude the required result. Conversely is easily. $\square$

Theorem 3.2. *Let $f_1(z) = z - \sum_{n=2}^{\infty} a_{n,1}z^n$ and $f_2(z) = z - \sum_{n=2}^{\infty} a_{n,2}z^n$ belongs to the class $\mathcal{W}_Q(a,b)$, then the weighted mean of f_1, f_2 in to $\mathcal{W}_Q(a,b)$.*

Proof. Let $H_t(z) = \frac{1}{2}(1+t)f_1(z) + \frac{1}{2}(1-t)f_2(z)$ we have

$$H_t(z) = z - \frac{1}{2}\sum_{n=2}^{\infty}((1+t)a_{n,1} + (1-t)a_{n,2})z^n$$

since f_1 and f_2 are in the class $\mathcal{W}_Q(a,b)$, so by theorem(2.1) we have

$$\sum_{n=2}^{\infty} \frac{a(n(Q+1)) + Q(1-b) + \dfrac{n+1}{n}}{((n-1)!)^2}[\frac{1}{2}(1+t)a_{n,1} + \frac{1}{2}(1-t)a_{n,2}]$$
$$\leq \frac{1}{2}(1+t)(1+Q(b-a-1)) + \frac{1}{2}(1-t)(1+Q(b-a-1))$$

Which the above expression is equal to $Q(b-a-1)+1$. So the condition of theorem(2.1) for $H_t(z)$ is established and therefore the proof is complete. $\square$

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Oral Presentation

NEVANLINNA THEORY AND SOME RELATED PROBLEMS

JAFAR SHAFFAF

Department of Mathematical Sciences, Shahid Beheshti University, P.O. Box 1983963113, Tehran, Iran.
j_Shaffaf@sbu.ac.ir

Abstract. This is an expository manuscript to introduce basics of Nevanlinna theory and some of its applications in complex function for the general audience with minimum back ground of complex analysis. We will see how the fundamental theorem of algebra for polynomials and the notion of their degree can be generalized naturally to entire and meromorphic functions. The aim of writing this note is to introduce this beautiful branch of complex analysis to students and the mathematical community around us.

1. Jensen Formula

The order of growth of an entire function is defined by $\rho_f = \inf\{\rho > 0 : |f(z)| \leq Ae^{b|z|^\rho}\}$, where A, B are positive constants. J. Hadamard proved that for a function of order ρ as in the above definition the degree of the polynomial in the Weirestrass theorem is at most ρ.
If we assume that the holomorphic function f has no zero on a disc of radius r around origin, then the function $\log f(z)$ is also holomorphic and well-defined on this disc and by applying the mean value property for this function we obtain :

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta$$

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Now if we assume that the function $f(z)$ has zeros $a_1, \cdots, a_n$ in the disc $D(0, r)$

Theorem 1.1. *(Jensen)- If f is a holomorphic function in the disc of radius r around origin and $f(0) \neq 0$ which has zeros $a_1, \cdots, a_n$contained in this disc then we have the following formula.*

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta - \sum_{j=1} \log|\frac{r}{a_j}|$$

.

Making use of the Jensen formula we have stitute the above in the

$$\int_0^r \frac{n(t,0)}{t}dt = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta - \log|f(0)|$$

Now if the order of growth of the function f according to the definition is ρ then by the above formula we can give an asymptotic bound for the number of zeros as the following.

$$n(r, 0) = O(r^\rho).$$

Also Borel generalized the above asymptotic for every complex number a except possibly for one complex number a as the following.

$$\limsup_{r\to\infty} \frac{\log n(r,a)}{\log r} = \rho,$$

so far we have seen that though entire functions is not easy to deal with as the realm of polynomials but we can still drive a good formula for the number of zeros asymtotically.

It was Rolf Nevanlinna to do this task for further development of the theory.

2. Birth of Nevanlinna Theory

The key idea of Nevanlinna is to use the Jensen formula for meromorphic functions by a slight modification that make it possible to apply this formula to these functions.

Let's define this real function $\log^+ x = \max(\log x, 0)$ (notice also that, this function play an important role to define the notion of height in nonarchemedian geometry which we do not want to deal with here). Using this function Nevanlinna introduced the following three functions to measure and analyze the behaviior of the function f. first define :

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta,$$

which is called the mean function or proximity function and it is in fact the average of the function f over the circle of radius r and measures how big the function is on the circle $|z| = r$. The second function is the following.

$$N(r, f) = \int_0^r \frac{n(t, f)}{t} dt,$$

which is called the Nevanlinna counting function and $n(t, f)$ is the number of poles of the function f in the disc $|z| \leq t$ and also we assume that $f(0) \neq 0, \infty$.
Now we try to rewrite the Jensen formula in terms of the function $\log^+ x$. Note that the important property of this function is $\log x = \log^+ x - \log^+ \frac{1}{x}$ and therefore we can obtain the following.

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta})} | d\theta$$
$$= m(r, f) - m(r, \frac{1}{f}).$$

and this says us that to have the Jensen formula for meromorphic functions as entire function we need to consider a similar sum to one that over zeros, but this new sum is over the poles of the meromorphic function f and this is the brilliant idea of Nevanlinna. Anyway if we apply the above formula and recall how we substitute the sum in Jensen formula by a definite integral, in the end we obtain the following new interpretation of Jensen formula.

$$\log |f(0)| = m(r, f) - m(r, \frac{1}{f}) + N(r, f) - N(r, \frac{1}{f}) \tag{2.1}$$

The important point about the above formula is that the left hand side is constant even when we tend r to infinity. Nevanlinna realized this fact and manage to define the growth of a meromorphic function as the following.
Nevanlinna characteristic function- $T(r, f) = m(r, f) + N(r, f)$.
We can write the formula (3.1) as the following.

$$T(r, \frac{1}{f}) = T(r, f) - \log |f(0)|, \tag{2.2}$$

and this is the point where Nevanlinna theory was born. In fact the first main theorem of Nevanlinna theory is rewriting the above formula as the following.

Theorem 2.1. *(First Main Theorem) -* $T(r, \frac{1}{f}) = T(r, f) + O(1)$*, where* $O(1)$ *is a bounded value.*

It seems that the above theorem was obtained very easily but it has great implications. In fact it says that the functions f likes the value 0 as much as the value ∞.
In fact write the above theorem for every value $a \in \mathbb{C}$ as the following

Theorem 2.2. *(First Main Theorem) -* $T(r, \frac{1}{f-a}) = T(r, f) + O(1)$ *as* $r \to \infty$.

Let the function f be a rational function, it can be written in the form $f(z) = \frac{p(z)}{q(z)}$, where $p(z)$ is a degree n polynomial and $q(z)$ is a degree m polynomial and $m \geq n$. By applying the above formula to the rational function $f(z)$ in the end we obtain:

$$T(r, f) = m \log r + O(1) = O(\log r).$$

Thus the rational functions has the growth condition $T(r, f) = O(\log r)$. Interestingly the converse of this fact is also true, it means the following. If a meromorphic function f has growth $T(r, f) = O(\log r)$ then this function should be a rational function. Therefore in terms of Nevanlinna theory we can generalize this fact for polynomials that they can be characterized by their order of growth (they have polynomially growth) to the realm of rational functions that says that the rational functions is the exact set of meromorphic functions with the logarithmic growth.

Theorem 2.3. *(Second Main Theorem) - Assume f is a meromorphic function and $a_1, \cdots, a_n$ is a set of complex numbers with $n > 2$ then we have the following inequality*

$$(n-2)T(r, f) \leq \sum_{i=1}^{n} N(r, a_i) + O(1).$$

The immediate consequence of the second main theorem of Nevanlinna theory is Picard's theorem. Assume that the meromorphic function f omits the three values a, b, and c. So we have $N(r, a) = N(r, b) = N(r, c) = 0$ for all r then by apply second main theorem for $n = 3$ we obtain:

$$T(r, f) \leq O(1),$$

and this is contradiction since the characteristic function $T(r, f)$ is always unbounded. Hence a meromorphic function can omit at most two values.
I will bring interesting applications of Nevanlinna theory (two main theorem in the above) to obtain some results in complex analysis in the next section. To understand more about the second main theorem and the structure of its error terms see [CY].

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CONFORMABLE STURM–LIOUVILLE PROBLEMS WITH TRANSMISSION CONDITIONS

MOHAMMAD SHAHRIARI

Department of Mathematics, Faculty of Sciences, University of Maragheh, Maragheh, Iran.
shahriari@maragheh.ac.ir

Abstract. In this manuscript, we study the Sturm–Liouville problem with conformable fractional differential operators of order α, $0.5 < \alpha \leq 1$ and finite number of interior discontinuous conditions. The asymptotic formulas of solutions, eigenvalues and eigenfunctions of the problem are calculated.

1. Introduction

Sturm–Liouville equation is one of the most important problems in mathematics, physics and engineering. This problem arises in modeling of many systems in vibration theory, quantum mechanics, hydrodynamic and so on [1]. The classical Sturm–Liouville equation is a second order ordinary differential equation of the following form:

$$y'' + (\lambda - q(x))y = 0, \quad 0 < x < \pi, \tag{1.1}$$

where $q(x)$ is the potential function and λ is a parameter. For equation (1.1) two boundary conditions at end points are considered. Equation (1.1) with boundary conditions are called Sturm–Liouville problems (SLP). Fractional Sturm-Liouville problems are different from those usually defined in this literature, i.e. the ordinary derivatives in a traditional Sturm-Liouville problem are replaced with fractional derivatives or derivatives of fractional

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order. These types of FSLP play a significant role in various areas of science, engineering, and mathematics [2, 3]. In this note, we study the asymptotic form of eigenvalues and characteristic functions of conformable fractional Sturm–Liouville problem (CFSLP).

2. Asymptotic form of solutions and eigenvalues

In this section, we give definition and some theorems of the conformable fractional(CF) derivative such that one can found in [4]. In what follows, we always take $D_x^\alpha = D^\alpha$.

Definition 2.1. For the function $f : [0, \infty) \to \mathbb{R}$, the CF derivative of f of order $\alpha \in (0, 1]$ defined by:

$$D^\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon},$$

for all $x > 0$, and

$$D^\alpha f(0) = \lim_{x \to 0^+} D^\alpha f(x).$$

If f is a differentiable function, then

$$D^\alpha f(x) = x^{1-\alpha} f'(x).$$

If $D^\alpha f(x_0)$ exists and is finite. Then the function f is α-differentiable at x_0.

Definition 2.2. The conformable integral of function f of order α is defined as:

$$J^\alpha f(x) = \int_0^x f(s) \mathrm{d}_\alpha s = \int_0^x s^{\alpha-1} f(s) \mathrm{d}s, \quad x > 0.$$

where, the integrals are in Riemann setting.

Let us consider the CFSLP

$$\ell_\alpha y := -D^\alpha D^\alpha y + qy = \lambda y \tag{2.1}$$

with boundary conditions

$$\begin{aligned} B_1(y) &:= D^\alpha y(0) + h\, y(0) = 0, \\ B_2(y) &:= D^\alpha y(\pi) + H\, y(\pi) = 0, \end{aligned} \tag{2.2}$$

and finite number of transmission conditions

$$\begin{aligned} U_i(y) &:= y(d_i+) - a_i y(d_i-) = 0, \\ V_i(y) &:= D^\alpha y(d_i+) - b_i D^\alpha y(d_i-) - c_i y(d_i-) = 0, \end{aligned} \tag{2.3}$$

for $i = 1, 2, \ldots, m-1$ and $\frac{1}{2} < \alpha \leq 1$. The parameters h, H and a_i, b_i, c_i, d_i are real numbers. We denote the problem (2.1)–(2.3) with $L_\alpha = L_\alpha(q(x); h; H; d_i)$. Consider the weighted inner product

$$\langle f, g \rangle_T := \int_0^\pi f(t) \overline{g(t)} w(t) \mathrm{d}_\alpha t,$$

where $f, g \in L_2^{\alpha}((0,\pi); w)$ and $w(t)$ is the weight function

$$w(t) = \begin{cases} 1, & 0 \leq t < d_1, \\ \frac{1}{a_1 b_1}, & d_1 < t < d_2, \\ \vdots & \\ \frac{1}{a_1 b_1 \cdots a_{m-1} b_{m-1}}, & d_{m-1} < t \leq \pi. \end{cases}$$

Note that $T := L_\alpha^2((0,\pi); w)$ is a Hilbert space with the norm $\|f\|_T = \langle f, f\rangle_T^{1/2}$. Let $A_\alpha : T \to T$ with domain

$$\operatorname{dom}(A_\alpha) = \left\{ f \in T \,\middle|\, \begin{array}{c} f, D^\alpha f \in AC\big(\cup_0^{m-1}(d_i, d_{i+1})\big), \\ \ell_\alpha f \in L_2^\alpha(0,\pi),\ U_i(f) = V_i(f) = 0 \end{array} \right\}$$

by

$$A_\alpha f = \ell_\alpha f, \quad f \in \operatorname{dom}(A_\alpha).$$

Suppose f and g are two solutions $\ell_\alpha f = \lambda f$, $\ell_\alpha g = \lambda g$ satisfying the jump conditions (2.3), the modified Wronskian

$$W_\alpha(f,g) = r(x)\big(f(x) D^\alpha g(x) - D^\alpha f(x) g(x)\big)$$

is constant for all $x \in [0, d_1) \cup_1^{m-2} (d_i, d_i + 1) \cup (d_{m-1}, \pi]$. Using the above formula $W_\alpha(f,g)(x) = W_\alpha(f,g)(x_0)$, for $x_0 \in [0,d) \cup (d,\pi]$. So, $W_\alpha(f,g)$ does not depend on x.

Lemma 2.3. *The operator A_α is self-adjoint on $L_2^\alpha((0,\pi); w)$.*

Let $u(x,\lambda)$ and $v(x,\lambda)$ be the solutions of (2.1) with the following initial conditions

$$u(0,\lambda) = 1,\ D^\alpha u(0,\lambda) = -h,\ v(\pi,\lambda) = 1,\ \text{and}\ D^\alpha v(\pi,\lambda) = -H,$$

and the jump conditions (2.3), respectively. The characteristic function is defined by

$$\Delta(\lambda) := W_\alpha(u(\lambda), v(\lambda)) = B_1(v(\lambda)) = -w(\pi) B_2(u(\lambda)).$$

Theorem 2.4. *Let $\lambda = \rho^2$ and $\tau := |\mathrm{Im}\rho|$. For CSLP (2.1)–(2.3) as $|\lambda| \to \infty$, the asymptotic forms of solutions and the characteristic function formula are in the following forms:*

$$u(x,\lambda)=\begin{cases}\cos\left(\frac{\rho}{\alpha}x^{\alpha}\right)+O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^{\alpha}\right)\right), & 0\le x<d_1,\\ \alpha_1\cos\left(\frac{\rho}{\alpha}x^{\alpha}\right)+\alpha_1'\cos\left(\frac{\rho}{\alpha}(x^{\alpha}-2d_1^{\alpha})\right)+O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^{\alpha}\right)\right), & d_1<x<d_2,\\ \alpha_1\alpha_2\cos\rho\left(\frac{\rho}{\alpha}x^{\alpha}\right)+\alpha_1'\alpha_2\cos\left(\frac{\rho}{\alpha}(x^{\alpha}-2d_1^{\alpha})\right)+\alpha_1\alpha_2'\cos\left(\frac{\rho}{\alpha}(x^{\alpha}-2d_2^{\alpha})\right) & \\ \quad+\alpha_1'\alpha_2'\cos\left(\frac{\rho}{\alpha}(x^{\alpha}+2d_1^{\alpha}-2d_2^{\alpha})\right)+O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^{\alpha}\right)\right), & d_2<x<d_3,\\ \quad\vdots & \\ \alpha_1\alpha_2\dots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}x^{\alpha}\right)+ & \\ \quad+\alpha_1'\alpha_2\dots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^{\alpha}-2d_1^{\alpha})\right)+\cdots & \\ \quad+\alpha_1\alpha_2\dots\alpha_{m-1}'\cos\left(\frac{\rho}{\alpha}(x^{\alpha}-2d_{m-1}^{\alpha})\right)+ & \\ \quad+\alpha_1'\alpha_2'\alpha_3...\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^{\alpha}+2d_1^{\alpha}-2d_2^{\alpha})\right)+\cdots & \\ \quad+\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^{\alpha}+2d_j^{\alpha}-2d_k^{\alpha})\right) & \\ \quad+\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_s'\dots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^{\alpha}-2d_j^{\alpha}+2d_k^{\alpha}-2d_s^{\alpha})\right)+\cdots & \\ \quad+\alpha_1'\alpha_2'\dots\alpha_{m-1}'\cos\left(\frac{\rho}{\alpha}(x^{\alpha}+2(-1)^{m-1}d_1^{\alpha}+2(-1)^{m-2}d_2^{\alpha}-2d_m^{\alpha})\right) & \\ \quad+O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^{\alpha}\right)\right), & d_{m-1}<x\le\pi,\end{cases}$$

where

$$\alpha_i=\frac{a_i+b_i}{2}\quad\textit{and}\quad\alpha_i'=\frac{a_i-b_i}{2},\quad i=1,2,\dots,m-1.$$

Also the similar asymptotic forms hold for the solutions $D^{\alpha}u, v$, and $D^{\alpha}v$. Moreover, we have

$$\begin{aligned}\Delta(\lambda)=&\rho w(\pi)\Big[\alpha_1\alpha_2\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}\pi^{\alpha}\right)+\alpha_1'\alpha_2\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^{\alpha}-2d_1^{\alpha})\right)+\cdots\\ &+\alpha_1\alpha_2\dots\alpha_{m-1}'\sin\left(\frac{\rho}{\alpha}(\pi^{\alpha}-2d_{m-1}^{\alpha})\right)+\alpha_1'\alpha_2'\alpha_3\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^{\alpha}+2d_1^{\alpha}-2d_2^{\alpha})\right)\\ &+\cdots+\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^{\alpha}+2d_j^{\alpha}-2d_k^{\alpha})\right)\\ &+\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_s'\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^{\alpha}-2d_j^{\alpha}+2d_k^{\alpha}-2d_s^{\alpha})\right)+\cdots\\ &+\alpha_1'\alpha_2'\dots\alpha_{m-1}'\sin\left(\frac{\rho}{\alpha}(\pi^{\alpha}+2(-1)^{m-1}d_1^{\alpha}+2(-1)^{m-2}d_2^{\alpha}-2d_m^{\alpha})\right)\Big]\\ &+O\left(\exp\left(\frac{\tau}{\alpha}\pi^{\alpha}\right)\right).\end{aligned}$$

Theorem 2.5. *Let $\lambda_n=\rho_n^2$ be the eigenvalues of the problem L_{α}, then we have the following asymptotic formula*

$$\rho_n=\alpha\pi^{1-\alpha}n+O(1)\qquad\textit{as } n\to\infty.$$

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Oral Presentation

INVERSE PROBLEM FOR DIRAC OPERATOR WITH DISCONTINUOUS CONDITIONS

MOHAMMAD SHAHRIARI

Department of Mathematics, Faculty of Sciences, University of Maragheh, Maragheh, Iran.
shahriari@maragheh.ac.ir

Abstract. In this manuscript, we study the inverse problem for Dirac operators with discontinuity conditions inside an interval. It is shown that the potential functions can be uniquely determined by a part of a set of values of eigenfunctions at an interior point and parts of one or two sets of eigenvalues.

1. Introduction

Let us consider the Dirac operator

$$\ell[y(x)] := By'(x) + \Omega(x)y(x) = \lambda y(x) \tag{1.1}$$

subject to the boundary conditions

$$\begin{aligned} U(y) &:= y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0, \\ V(y) &:= y_1(\pi)\cos\beta + y_2(\pi)\sin\beta = 0, \end{aligned} \tag{1.2}$$

and the jump conditions

$$C(y) := y(d+0) = Ay(d-0), \tag{1.3}$$

where $x \in I := [0, d) \cup (d, \pi]$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}$,

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$y(x) = \begin{pmatrix} y_1(x), & y_2(x) \end{pmatrix}^T$, and $A = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$.

In this paper, the functions $p(x)$ and $q(x)$ are real valued in $L_2(0, \pi)$, $a \in \mathbb{R} - \{0\}$, α, $\beta \in [0, \pi)$ and λ is a spectral parameter. For simplicity we use $L = L(\Omega(x); \alpha; \beta; d)$ for the above system of equation. It is easy to see that the operator L is a self-adjoint operator. Indeed the operator L has a discrete spectrum consisting simple and real eigenvalues λ_n, for $n \in \mathbb{Z}$.

In the paper [1], Amirov study the direct and inverse problems for Dirac operators with discontinuities inside an interval. Furthermore, direct or inverse spectral problems for Dirac operators were extensively studied in [4], and the references therein. In this manuscript, we study the inverse problem for Dirac differential operators with discontinuity conditions. It is shown that the potential functions can be uniquely determined by a part of a set of values of eigenfunctions at an interior point and parts of one or two sets of eigenvalues.

2. Preliminaries

Let the functions $u(., \lambda) : I \to \mathbb{R}^2$ be

$$\begin{aligned} &Bu'(x) + \Omega(x)u(x) = \lambda u(x) \\ &u_1(0) = \sin\alpha, \ u_2(0) = -\cos\alpha. \end{aligned}$$

with the jump conditions (1.3) where $u(x, \lambda) = (u_1(x, \lambda), u_2(x, \lambda))^T$. It is shown in [2, 3] that there exit kernels $K(x, t) = (K_{ij}(x, t)^2_{i,j=1})$ with entire continuously differentiable on $0 \le t \le x < d$ such that the solution $u(x, \lambda)$ is

$$u(x, \lambda) = u_\circ(x, \lambda) + \int_0^x K(x, t) u_\circ(t, \lambda) dt \tag{2.1}$$

Here $u_\circ(x, \lambda) = \begin{pmatrix} u_{\circ 1}(x, \lambda), & u_{\circ 2}(x, \lambda) \end{pmatrix}^T$. It is easy to check that the following functions are solutions of (1.1) with $\Omega(x) = 0$,

$$u_{\circ 1}(x, \lambda) = \begin{cases} \sin(\lambda x + \alpha), & 0 \le x < d, \\ a^+ \sin(\lambda x + \alpha) + a^- \sin(\lambda(2d - x) + \alpha), & d < x \le \pi. \end{cases}$$

$$u_{\circ 2}(x, \lambda) = \begin{cases} -\cos(\lambda x + \alpha), & 0 \le x < d, \\ -a^+ \cos(\lambda x + \alpha) + a^- \cos(\lambda(2d - x) + \alpha), & d < x \le \pi. \end{cases}$$

where $a^+ = \frac{1}{2}\left(a + \frac{1}{a}\right)$, and $a^- = \frac{1}{2}\left(a - \frac{1}{a}\right)$. The characteristic function for $\begin{pmatrix} u_{\circ 1}(x, \lambda), & u_{\circ 2}(x, \lambda) \end{pmatrix}^T$ is

$$\Delta_\circ(\lambda) := a^+ \sin(\lambda\pi + \alpha - \beta) + a^- \sin(\lambda(2d - \pi) + \alpha + \beta).$$

The roots of the entire function $\Delta_\circ(\lambda)$ are simple and real. The roots of $\Delta_\circ(\lambda)$ is

$$\lambda_n^\circ = n + M_n$$

such that $\sup_n M_n < M < \infty$.

Suppose $v(x,\lambda) = \begin{pmatrix} v_1(x,\lambda), & v_2(x,\lambda) \end{pmatrix}^T$ be the solution of (1.1) with the initial conditions

$$v(\pi,\lambda) = \begin{pmatrix} \sin\beta, -\cos\beta \end{pmatrix}^T.$$

By changing x to $\pi - x$ one can obtain the similar form of (1.2) for the solution $v(x,\lambda)$ on the interval $(d,\pi]$.

We define the characteristic function for the operator L of the form

$$\Delta(\lambda) := \langle u(x,\lambda), v(x,\lambda)\rangle = \int_0^\pi (u_1\bar{v}_1 + u_2\bar{v}_2)dx$$

The characteristic function $\Delta(\lambda)$ is independent of x. It flows from (2.1) and the same form of (2.1) for $v(x,\lambda)$ on the jump point $x = d$, so we have

$$\Delta(\lambda) = \Delta_\circ(\lambda) + O\left(\frac{\exp(|\tau|\pi)}{\lambda}\right)$$

where $\tau = |\mathrm{Im}\lambda|$. The zeros of $\Delta(\lambda)$ are the eigenvalues of L and hence it has only simple and real zeros λ_n. We denote by $y_n(x) = (y_{n,1}(x), y_{n,2}(x))^T$, $n \in \mathbb{Z}$, the corresponding eigenfunctions.

Theorem 2.1. *The corresponding eigenvalues $\{\lambda_n\}$ of the boundary value problem L admit the following asymptotic form as $n \to \infty$:*

$$\lambda_n = n + O(1).$$

Suppose $v(x,\lambda) = \begin{pmatrix} v_1(x,\lambda), & v_2(x,\lambda) \end{pmatrix}^T$ be the solution of (1.1) with the initial conditions

$$v(\pi,\lambda) = \begin{pmatrix} \sin\beta, -\cos\beta \end{pmatrix}^T.$$

By changing x to $\pi - x$ one can obtain the similar form of (1.2) for the solution $v(x,\lambda)$ on the interval $(d,\pi]$. Define the characteristic function for the operator L of the form

$$\Delta(\lambda) := \langle u(x,\lambda), v(x,\lambda)\rangle.$$

The characteristic function $\Delta(\lambda)$ is independent of x. It flows from (2.1) and the same form of (2.1) for $v(x,\lambda)$ on the jump point $x = d$, so we have

$$\Delta(\lambda) = \Delta_\circ(\lambda) + O\left(\frac{\exp(|\tau|\pi)}{\lambda}\right)$$

where $\tau = |\mathrm{Im}\lambda|$. The zeros of $\Delta(\lambda)$ are the eigenvalues of L and hence it has only simple and real zeros λ_n. We denote by $y_n(x) = (y_{n,1}(x), y_{n,2}(x))^T$, $n \in \mathbb{Z}$, the corresponding eigenfunctions.

Theorem 2.2. *The corresponding eigenvalues $\{\lambda_n\}$ of the boundary value problem L admit the following asymptotic form as $n \to \infty$:*

$$\lambda_n = n + O(1).$$

3. INVERSE PROBLEM

Let us introduce a second Dirac operator $\tilde{L} = L(\tilde{\Omega}(x); \alpha; \beta; d)$ here

$$\tilde{\Omega}(x) = \begin{pmatrix} \tilde{p}(x) & \tilde{q}(x) \\ \tilde{q}(x) & -\tilde{p}(x) \end{pmatrix}$$

with a real valued function $\tilde{p}(x),\ \tilde{q}(x) \in L^2(0,\ \pi)$. The eigenvalues and the corresponding eigenfunctions of $\tilde{L}$ are denoted by $\tilde{\lambda}_n$ and $\tilde{y}_n(x) = (\tilde{y}_{n,1}(x), \tilde{y}_{n,2}(x))^T (n \in \mathbb{Z})$, respectively.

Theorem 3.1. *If $\lambda_n = \tilde{\lambda}_n,\ \langle y_n, \tilde{y}_n \rangle_{d-0} = 0$ for any $n \in \mathbb{Z}$ and $d \le \frac{\pi}{2}$ then $p(x) = \tilde{p}(x),\ q(x) = \tilde{q}(x)$ a.e on the $[0, d)$.*

Remark 3.2. We can easily obtain if y, z be the solution of (1.1) and satisfy the jump conditions (1.3) and $\langle y, z \rangle_{(a-0)} = 0$ then $\langle y, z \rangle_{(a+0)} = 0$

Corollary 3.3. *Let $d \in (\frac{\pi}{2}, \pi)$ be a jump point. Let $\lambda_n = \tilde{\lambda}_n$, and $\langle y_n, \tilde{y}_n \rangle_{(d-0)} = 0$, for each $n \in \mathbb{Z}$. Then $\Omega(x) = \tilde{\Omega}(x)$ almost everywhere on $(d, \pi]$.*

Remark 3.4. For $d = \frac{\pi}{2}$ from Theorems 3.1 and 3.3, we get $\Omega(x) = \tilde{\Omega}(x)$ almost everywhere on $[0, \pi]$.

Theorem 3.5. *Let $d \in (\frac{\pi}{2}, \pi]$ be a jump point and $\sigma > \frac{2a}{\pi} - 1$. Let*

$$\lambda_n = \tilde{\lambda}_n,\ \mu_{l(n)} = \tilde{\mu}_{l(n)},\ \text{and } \langle y_n, \tilde{y}_n \rangle_{(d-0)} = 0,$$

for each $n \in \mathbb{Z}$. Then $\Omega(x) = \tilde{\Omega}(x)$ almost everywhere on $[0, d) \cup (d, \pi]$.

Corollary 3.6. *Let $d \in (0, \frac{\pi}{2}]$ be a jump point, fix $b \in (0, d]$ and $\sigma_1 > \frac{2b}{\pi}$. Let $\lambda_{m(n)} = \tilde{\lambda}_{m(n)}$ $\langle y_n, \tilde{y}_n \rangle_{d-0} = 0$, for each $n \in \mathbb{Z}$. Then $\Omega(x) = \tilde{\Omega}(x)$ almost everywhere on $[0, \pi]$.*

Let $r(n)$ be a subsequence of natural numbers such that

$$r(n) = \frac{n}{\sigma_2}(1 + \epsilon_{2n}), 0 < \sigma_2 \le 1, \epsilon_{2n} \to 0$$

Corollary 3.7. *Let $d \in (\frac{\pi}{2}, \pi)$ be a jump point, fix $\sigma > \frac{2d}{\pi} - 1$ and $\sigma_2 > 2 - \frac{2d}{\pi}$. If for each $n \in \mathbb{N}$*

$$\lambda_n = \tilde{\lambda}_n,\ \mu_{l(n)} = \tilde{\mu}_{l(n)},\ \langle y_{r(n)}, \tilde{y}_{r(n)} \rangle_{(d-0)} = 0,$$

then $\Omega(x) = \tilde{\Omega}(x)$ almost everywhere on $[0, \pi]$.

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ON THE EXISTENCE RESULTS FOR A CLASS OF NONLOCAL ELLIPTIC SYSTEMS

S. SHAKERI

Department of Mathematics, Ayatollah Amoli Branch, Islamic Azad University, Amol, Iran
s.shakeri@iauamol.ac.ir

Abstract. In this paper, we prove some existence results on positive solution for a class of nonlocal elliptic systems in bounded domains. We employ the method of sub-super solutions to establish our results.

1. Introduction

In this article, we mainly consider the existence of a positive solution of the following singular elliptic system

$$\begin{cases} -M_1\Big(\displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = \lambda a(x) f(v) - u^{-\alpha} & \text{in } \Omega, \\ -M_2\Big(\displaystyle\int_\Omega |\nabla v|^p dx\Big)\Delta_p v = \lambda b(x) g(u) - v^{-\alpha} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a positive parameter, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$,$M_i : R_0^+ \to R^+$, $i = 1, 2$, are two continuous and increasing functions, $\Omega \subset R^n$ some for $n > 1$, is a bounded domain with smooth boundary $\partial\Omega$, $0 < \alpha < 1$, and $f, g : [0, \infty] \to R$ are continuous, nondecreasing functions which are asymptotically p-linear at ∞. We prove the existence of a positive solution for a certain range of λ.

We consider problem (1.1) under the following assumptions.

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(H_1) There exist $\sigma_1 > 0, k_1 > 0$ and $s_1 > 1$ such that

$$f(s) \geq \sigma_1 s^{p-1} - k_1$$

for every $s \in [0, s_1]$
and that there exist $\sigma_2 > 0, k_2 > 0$ and $s_2 > 1$ such that

$$g(s) \geq \sigma_2 s^{p-1} - k_2$$

for every $s \in [0, s_2]$,

(H_2) For all $M > 0$, $\lim_{s\to+\infty} \frac{f(M[g(s)]^{\frac{1}{p-1}})}{s^{p-1}} = \sigma$ for some $\sigma > 0$.

(H_3) $a, b : \overline{\Omega} \to (0, \infty)$ are continuous functions such that $a_1 = \min_{x\in\overline{\Omega}} a(x)$, $b_1 = \min_{x\in\overline{\Omega}} b(x)$, $a_2 = \max_{x\in\overline{\Omega}} a(x)$ and $b_2 = \max_{x\in\overline{\Omega}} b(x)$.

(H_4) There exists $\tau \in \mathbb{R}$ such that for each $M > 0, f(Ms) \leq M^\tau f(s)$ for $s \gg 1$.

(H_5) $M_i : R_0^+ \to R^+$, $i = 1, 2$, are two continuous and increasing functions and $0 < m_i \leq M_i(t) \leq m_{i,\infty}$ for all $t \in R_0^+$, where $R_0^+ := [0, +\infty)$.

System (1.1) is related to the stationary problem of a model introduced by Kirchhoff [4].Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where ρ, ρ_0, h, E are all constants. This equation is an extension of the classical D'Alembert's wave equation. A distinguishing feature of equation (1.2) is that the equations a nonlocal coefficient $\frac{P_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx$ which depends on the average $\frac{1}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx$; hence the equation is no longer a pointwise identity. Nonlocal problems can be used for modeling, for example, physical and biological systems for which u describes a process which depends on the average of itself, such as the population density. Let $F(u) := \lambda a(x)f(u) - u^{-\alpha}$. The case when $F(0) < 0$ (and finite) is referred to in the literature as a semipositone problem. Here we consider the more challenging case when $\lim_{u\to 0^+} F(u) = -\infty$, which has received attention very recently and is referred to as an infinite semipositone problem. In this note, we study the existence of positive solutions for Kirchhoff type system (1.1). Our result improves the previous one introduced by G.Afrouzi et al. [1] in which $M_1(t) = M_2(t) \equiv 1$. We shall establish our an existence result via the method of sub and supersolutions.

Now, we give the definitions of sub- and super-solutions of (1.1).

Definition 1.1. We say that (ψ_1, ψ_2) (*resp.* (z_1, z_2)) in $(W^{1,p}(\Omega)\cap C(\overline{\Omega}), W^{1,p}(\Omega)\cap C(\overline{\Omega}))$ are called a subsolution (*resp.* a supersolution) of (1.1), if ψ_i $(i = 1, 2)$

satisfy

$$\begin{cases} M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\int_\Omega |\nabla\psi_1(x)|^{p-2}\nabla\psi_1\cdot\nabla w_1 dx \le \int_\Omega \Big(\lambda a(x)f(\psi_2)-\psi_1^{-\alpha}\Big)w_1(x)dx \\ M_2\Big(\int_\Omega |\nabla\psi_2|^p dx\Big)\int_\Omega |\nabla\psi_2(x)|^{p-2}\nabla\psi_2(x)\cdot\nabla w_2 dx \le \int_\Omega \Big(\lambda b(x)g(\psi_1)-\psi_2^{-\alpha}\Big)w_2(x)dx \\ \psi_1,\psi_2>0 \quad \text{in } \Omega, \\ \psi_1=\psi_2=0 \quad \text{on } \partial\Omega \end{cases} \tag{1.3}$$

$$\Bigg(\textit{resp. } z_i \ (i=1,2) \text{ satisfy:}$$

$$\begin{cases} M_1\Big(\int_\Omega |\nabla z_1|^p dx\Big)\int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w_1(x) dx \ge \int_\Omega \Big(\lambda a(x)f(z_2)-z_1^{-\alpha}\Big)w_1(x)dx \\ M_2\Big(\int_\Omega |\nabla z_2|^p dx\Big)\int_\Omega |\nabla z_2|^{p-2}\nabla z_2\cdot\nabla w_2(x) dx \ge \int_\Omega \Big(\lambda b(x)g(z_1)-z_2^{-\alpha}\Big)w_2(x)dx \\ z_1,z_2>0 \quad \text{in } \Omega, \\ z_1=z_2=0 \quad \text{on } \partial\Omega\Bigg) \end{cases} \tag{1.4}$$

for all non-negative test functions $w_i (i=1,2)\in W$, where $W=\{\xi\in C_0^\infty(\Omega):\xi\ge 0 \text{ in } \Omega\}$.

2. Main result

In this paper, we denote $W_0^{1,r}(\Omega)$, the completion of $C_0^\infty(\Omega)$, with respect to the norm

$$\|u\|_r=\left(\int_\Omega |\nabla u|^r\,dx\right)^{\frac{1}{r}}.$$

In order to precisely state our main result we first consider the following eigenvalue problem for the r-Laplace operator $-\Delta_r u$, see [5]:

$$\begin{cases} -\Delta_r u=\lambda|u|^{r-2}u \quad \text{in } \Omega, \\ u=0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Let $\phi_{1,r}\in C^1(\overline{\Omega})$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,r}$ of (2.1) such that $\phi_{1,r}>0$ in Ω and $\|\phi_{1,r}\|_\infty=1$. It can be shown that $\frac{\partial\phi_{1,r}}{\partial\eta}<0$ on $\partial\Omega$ and hence, depending on Ω, there exist positive constants m,δ,σ such that

$$\begin{cases} |\nabla\phi_{1,r}|^r-\lambda_{1,r}\phi_{1,r}^r\ge m \quad \text{on } \overline{\Omega}_\delta, \\ \phi_{1,r}\ge\sigma \quad \text{on } x\in\Omega\backslash\overline{\Omega}_\delta, \end{cases} \tag{2.2}$$

where $\overline{\Omega}_\delta:=\{x\in\Omega:\ d(x,\partial\Omega)\le\delta\}$.

We will also consider the unique solution $e_r\in W_0^{1,r}(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta_r e_r=1 \quad \text{in } \Omega, \\ e_r=0 \quad \text{on } \partial\Omega \end{cases} \tag{2.3}$$

to discuss our result. It is known that $e_r>0$ in Ω and $\frac{\partial e_r}{\partial\eta}<0$ on $\partial\Omega$.

Our main result is given by the following theorem.

Theorem 2.1. *Assume the conditions* $(H_1)-(H_5)$ *are satisfied. Then there exist positive constants* $s_0^*(\sigma,\Omega)$, $J^*(\Omega)$, λ_*, *and* $\lambda_{**}(>\lambda_*)$ *such that if min* $\{s_1, s_2\} \geq s_0^*$ *and* $\frac{min\{a_1,b_1\}min\{\sigma_1,\sigma_2\}}{(\sigma)^{\frac{p-1}{p-1+\tau}}} \geq J^*$, *problem (1.1) has a positive solution for* $\lambda \in [\lambda_*, \lambda_{**}]$.

A key role in our arguments will be played by the following auxiliary result. Its Proof is similar to those presented in [2], the reader can consult further the papers [3].

Lemma 2.2. *Assume that* $M : R^+ \to R^+$ *is a continuous and increasing function satisfying*

$$M(t) \geq M_0 > 0 \textit{ for all } t \in R^+.$$

If the functions $u, v \in W_0^{1,p}(\Omega)$ *satisfies*

$$M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx \leq M\left(\int_\Omega |\nabla v|^p\,dx\right)\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx \tag{2.4}$$

for all $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geq 0$, *then* $u \leq v$ *in* Ω.

From Lemma (2.2) we can establish the basic principle of the sub- and supersolutions method for nonlocal systems. Indeed, we consider the following nonlocal system

$$\begin{cases} -M_1\Big(\displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = h(x,u,v) \text{ in } \Omega, \\ -M_2\Big(\displaystyle\int_\Omega |\nabla v|^q dx\Big)\Delta_q v = k(x,u,v) \text{ in } \Omega, \\ u = v = 0 \text{ on } x \in \partial\Omega, \end{cases} \tag{2.5}$$

where Ω is a bounded smooth domain of R^N and $h, k : \overline{\Omega} \times R \times R \to R$ satisfy the following conditions

(HK1) $h(x,s,t)$ and $k(x,s,t)$ are Carathéodory functions and they are bounded if s, t belong to bounded sets.

(KH2) There exists a function $g : R \to R$ being continuous, nondecreasing, with $g(0) = 0$, $0 \leq g(s) \leq C(1 + |s|^{\min\{p,q\}-1})$ for some $C > 0$, and applications $s \mapsto h(x,s,t) + g(s)$ and $t \mapsto k(x,s,t) + g(t)$ are nondecreasing, for a.e. $x \in \Omega$.

If $u, v \in L^\infty(\Omega)$, with $u(x) \leq v(x)$ for a.e. $x \in \Omega$, we denote by $[u, v]$ the set $\{w \in L^\infty(\Omega) :\ u(x) \leq w(x) \leq v(x) \text{ for a.e. } x \in \Omega\}$.

Proposition 2.3. *Let* $M_1, M_2 : R_0^+ \to R^+$ *be two functions satisfying the condition* ***(H1)***. *Assume that the functions* h, k *satisfy the conditions (HK1) and (HK2). Assume that* $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$, *are respectively, a weak subsolution and a weak supersolution of system (2.5) with* $\underline{u}(x) \leq \overline{u}(x)$ *and* $\underline{v}(x) \leq \overline{v}(x)$ *for a.e.* $x \in \Omega$. *Then there exists a minimal* (u_*, v_*) *(and, respectively, a maximal* (u^*, v^*)*) weak solution for system (2.5) in the set* $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. *In* *particular, every weak solution* $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ *of system* (2.5) *satisfies* $u_*(x) \le u(x) \le u^*(x)$ *and* $v_*(x) \le v(x) \le v^*(x)$ *for a.e.* $x \in \Omega$.

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EFFECTIVE IMPLEMENTATION OF LEGENDRE POLYNOMIALS IN PRICING DISCRETELY MONITORED DOUBLE BARRIER OPTION

AMIRHOSSEIN SOBHANI* AND MOHAMMAD HOSSEIN BEHESHTI

Department of Mathematics, Statistics and Computer Science, Semnan University, Semnan, Iran
Faculty of Medicine, Tehran Medical Sciences, Islamic Azad University, Tehran, Iran

Abstract. In the present paper, Legendre polynomials are effectively implemented in pricing discrete double barrier options which are commonly done through recursive solving Black-Scholes PDEs in the monitoring intervals. By using orthogonal projection based on Legendre polynomials, we could obtain an operational matrix to approximate the price of the option.

1. Introduction

A knock-out double barrier option is an option that is deactivated when the price of the underlying asset touches each of the two predetermined barriers before the expiry date at monitoring dates. Various approaches have been proposed for pricing barrier options. An analytical method is derived by Fusai et al. in [1] based on z-transform. The method of finite element is used by Golbabai et al.[2]. Milev and Tagliani presented a numerical algorithm for pricing discrete double barrier options [3]. Farnoosh et al. [4, 5] provide methods for pricing discretely monitored (single or double) barrier options that work even for the case of time-dependent parameters.

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* Speaker.

In this paper, Legendre Polynomials is effectively implemented as an orthogonal basis for the projection method that causes to operational matrix form. Computational time is almost fixed and not affected by the number of monitoring dates. According to the Black-Scholes framework, the price of discretely monitored double barrier call option as a function of stock price s at time $t \in (t_m, t_{m+1})$, namely $\mathcal{C}(s,t,m)$, is obtained from forward solving the following partial differential equations with the initial conditions[6]:

$$-\frac{\partial \mathcal{C}}{\partial t} + \mu s \frac{\partial \mathcal{C}}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \mathcal{C}}{\partial s^2} - r\mathcal{C} = 0, \tag{1.1}$$
$$\mathcal{C}(s,t_0,0) = (s-E)\,\mathbf{1}_{(\max(E,L)\le s\le U)}\,,$$
$$\mathcal{C}(s,t_m,m) = \mathcal{C}(s,t_m,m-1)\,\mathbf{1}_{(L\le s\le U)};\;\; m=1,2,...,M-1\;.$$

The constant coefficients μ and σ are risk-free rate and volatility respectively. Also, the constants E, L, and U are exercise price, lower and upper barrier respectively. In the following, two changes of variables are performed. At first, the function $P(z,t,m)$ is defined as $P(z,t,m) := \mathcal{C}(s,t,m)$ where $z = \ln\left(\frac{s}{L}\right)$, $E^* = \ln\left(\frac{E}{L}\right)$, $\mu^* = \mu - \frac{\sigma^2}{2}$, $U^* = \ln\left(\frac{U}{L}\right)$, $\delta = \max\{E^*, 0\}$. Then the partial differential equation (1.1) and its initial conditions are changed into:

$$-P_t + \mu^* P_z + \frac{\sigma^2}{2} P_{zz} - \mu P = 0, \tag{1.2}$$
$$P(z,t_0,0) = L\left(e^z - e^{E^*}\right)\mathbf{1}_{(\delta\le z\le U^*)},$$
$$P(z,t_m,m) = P(z,t_m,m-1)\,\mathbf{1}_{(0\le z\le U^*)}\,;\;\; m=1,2,...,M-1\;.$$

As a second step, the following transformation is applied:

$$P(z,t_m,m) = e^{\alpha z+\beta t} g(z,t,m),$$

where $\alpha = -\frac{\mu^*}{\sigma^2}$; $c^2 = \frac{\sigma^2}{2}$; $\beta = \alpha\mu^* + \alpha^2\frac{\sigma^2}{2} - \mu$.
Therefore, the partial differential equation (1.2) and its initial conditions are led to:

$$-g_t + c^2 g_{zz} = 0, \tag{1.3}$$
$$g(z,t_0,0) = Le^{-\alpha z}\left(e^z - e^{E^*}\right)\mathbf{1}_{(\delta\le z\le U^*)},$$
$$g(z,t_m,m) = g(z,t_m,m-1)\,\mathbf{1}_{(0\le z\le U^*)};\; m=1,...,M-1.$$

The resulting expressions in (1.3) are known as heat equations. Analytical solutions to the heat equations at the monitoring dates of equal distances $\tau = \frac{T}{M}$ or equivalently $t_m = m\tau$, are denoted by $f_m(z) := g(z,t_m,m-1)$ and evaluated as follows, see e.g [7];

$$f_0(z) = Le^{-\alpha z}\left(e^z - e^{E^*}\right)\mathbf{1}_{(\delta\le z\le U^*)}, \tag{1.4}$$

$$f_m(z) = \mathcal{K}(f_{m-1}(z)),\;\; m=2,3,...,M-1, \tag{1.5}$$

where the compact operator $\mathcal{K} : L^2([0,U^*]) \to L^2([0,U^*])$ is defined as follows:

$$\mathcal{K}(f)(z) := \int_0^{U^*} \frac{1}{\sqrt{4\pi c^2\tau}} e^{-\frac{(z-\xi)^2}{4c^2\tau}} f(\xi)d\xi. \tag{1.6}$$

According to the above stages, the price of the knock-out discrete double barrier European call Option at expiry date T is evaluated by the following formula:

$$\mathcal{C}\left(s_0, T, M-1\right) = e^{(\alpha z_0+\beta T)} f_{M-1}\left(z_0\right), \tag{1.7}$$

where $z_0 = \ln\left(\frac{s_0}{L}\right)$.

2. Implementation of Legendre Polynomials

$$p_i(x) = xp_{i-1}(x) + \left(\frac{i}{i+1}\right)\left(xp_{i-1}(x) - p_{i-2}(x)\right),$$

where $p_0(x) = 1$, and $p_1(x) = x$. The $\{p_i(x)\}_{i=0}^{\infty}$ is an orthogonal basis for $L^2[-1,1]$. Now, we define $\tilde{p}_i(x) := \sqrt{\frac{2i+1}{U^*}} p_i\left(\frac{U^*}{2}x + \frac{U^*}{2}\right)$. These functions constitute an orthonormal basis for $L^2[0, U^*]$. Consider $\Pi_n = span\left\{\tilde{p}_i(x)\right\}_{i=0}^{n}$ be the space of all polynomials with degrees less than or equal to n and also $P_n : L^2[0, U^*] \rightarrow \Pi_n$ be orthogonal projection operator, that is defined as follows:

$$\forall f \in L^2[0, U^*] \quad P_n\left(f\right) = \sum_{i=0}^{n} \langle f, \tilde{p}_i(x)\rangle \tilde{p}_i(x), \tag{2.1}$$

where $\langle\ .\ ,\ .\ \rangle$ indicates the usual inner product.

Now, we define $\tilde{f}_{m,n} = P_n\mathcal{K}\left(\tilde{f}_{m-1,n}\right) = \left(P_n\mathcal{K}\right)^m\left(f_0\right)$, $m \geq 2$ where $(P_n\mathcal{K})(f) = P_n\left(\mathcal{K}(f)\right)$. Since the continuous projection operators P_n converge pointwise to identity operator I, then operator $P_n\mathcal{K}$ is also a compact operator and it could be shown that

$$\lim_{n\to\infty} \left\|(P_n\mathcal{K})^m - \mathcal{K}^m\right\| = 0. \tag{2.2}$$

Since, $\tilde{f}_{m,n} \in \Pi_n$ for $m \geq 1$, we can write

$$\tilde{f}_{m,n} = \sum_{i=0}^{n} a_{mi}\tilde{p}_i(z) = \Phi_n(x)F_m,$$

where $F_m = [a_{m0}, a_{m1}, \cdots, a_{m2^j}]'$ and $\Phi_n = [\tilde{p}_0(x), \tilde{p}_1(x), \cdots, \tilde{p}_n(x)]'$. So we obtain

$$\tilde{f}_{m,n} = (P_n\mathcal{K})^{m-1}\left(\tilde{f}_{1,n}\right). \tag{2.3}$$

Because Π_n is a finite-dimensional linear space, so the linear operator $P_n\mathcal{K}$ on Π_n could be considered as a $(n+1)\times(n+1)$ matrix K. Consequently, equation 2.3 can be written as the following matrix operator form:

$$\tilde{f}_{m,n} = \Phi_n' K^{m-1} F_1. \tag{2.4}$$

For computation of the option price by 2.4, it is enough to calculate the matrix operator K and the vector F_1:

$$F_1 = [a_{10}, a_{11}, \cdots, a_{1n}]', \quad K = (k_{ij})_{(n+1)\times(n+1)},$$

$$a_{1i} = \int_0^{U^*} \int_\delta^{U^*} \tilde{p}_i(\eta)\kappa(\eta-\xi,\tau)f_0(\xi)d\xi d\eta \ , \ 0 \le i \le n,$$

$$k_{ij} = \int_0^{U^*} \int_0^{U^*} \tilde{p}_i(\eta)\tilde{p}_j(\xi)\kappa(\eta-\xi,\tau)d\xi d\eta \ ,$$

where $\kappa(z,t) = \frac{1}{\sqrt{4\pi c^2 t}} e^{-\frac{z^2}{4c^2 t}}$. The matrix form of relation 2.4 implies that the computational time of the presented algorithm be nearly fixed when monitoring dates increase. The complexity of our algorithm is $\mathcal{O}(n^2)$ that does not depend on the number of monitoring dates.

3. Numerical Result

Here, price of a double knock-out barrier option with $T = 0.5$, $\mu = 0.05$, $\sigma = 0.25$, $s_0 = 100$ $E = 100$, $U = 120$ and different level of lower barrier L is approximated by presented method. The numerical results are reported and compared with some other ones. The CPU time of the Presented method does not increase significantly when the number of monitoring dates increases.

M	L	Legendre($n = 16$)	Quad-K30	AMM-8	Benchmark
	80	2.4499	2.4499	2.4499	2.4499
	90	2.2028	2.2028	2.2027	2.2028
5	95	1.6831	1.6831	1.6830	1.6831
	99	1.0811	1.0811	1.0811	1.0811
	99.9	0.9432	0.9432	0.9433	0.9432
CPU		0.52 s			
	80	1.9420	1.9420	1.9419	1.9420
	90	1.5354	1.5354	1.5353	1.5354
25	95	0.8668	0.8668	0.8668	0.8668
	99	0.2931	0.2931	0.2932	0.2931
	99.9	0.2023	0.2023	0.2024	0.2023
CPU		0.54 s			

Table 1: Double knock-out barrier option: $T = 0.5$, $\mu = 0.05$, $\sigma = 0.25$, $s_0 = 100$, $E = 100$.

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Oral Presentation $\qquad$ $*$ **: Speaker**

MULTIPLICITY OF WEAK SOLUTIONS FOR AN ANISOTROPIC ELLIPTIC SYSTEM

T. SOLTANI*AND A. RAZANI

Department of Pure Mathematics, Imam Khomeini International University, Qazvin 34194, Iran
t.soltani@edu.ikiu.ac.ir; razani@sci.ikiu.ac.ir

ABSTRACT. Here, by using variational methods, the multiplicity of weak solutions for a system of problems including the anisotropic $\overrightarrow{p}(x)$-Laplacian operator is proved.

1. INTRODUCTION

Anisotropic $\overrightarrow{p}-$Laplacian operator

$$\Delta_{\overrightarrow{p}(x)}u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\frac{\partial u}{\partial x_i}|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right),$$

$\overrightarrow{p} = (p_1, \cdots, p_N)$, with a complex structure that behaves differently in different directions of space, has been the focus of many authors in recent years [**?**, **?**]. This operator is used in equations that descriptions electromagnetic fields, the plasma physics and elastic mechanics.

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In this paper, using variational methods, we examine the existence and multiplicity of weak solutions for anisotropic system

$$\begin{cases} -\Delta_{\overrightarrow{p}(x)}u+\sum_{i=1}^{N}a_1(x)|u|^{p_i(x)-2}u=\lambda F_u(x,u,v)+\mu G_u(x,u,v) & \text{in } \Omega,\\ -\Delta_{\overrightarrow{p}(x)}v+\sum_{i=1}^{N}a_2(x)|v|^{p_i(x)-2}v=\lambda F_v(x,u,v)+\mu G_v(x,u,v) & \text{in } \Omega,\\ \frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0 & \text{On } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$, is a non-empty bounded open set with a boundary $\partial\Omega$ of class C^1, ν is the outer unit normal to $\partial\Omega$. $\overrightarrow{p}=(p_1,\cdots,p_N)$ where for $i=1,\cdots,N$, p_is are continuous functions on Ω with $p_i(x)\geq 2$ for all $x\in\Omega$. Also λ,μ are positive parameters, F_ξ, G_ξ denote the partial derivative of F, G with respect to ξ and $F(x,.,.), G(x,.,.)$ are continuously differentiable in $\mathbb{R}^2$ for a.e. $x\in\Omega$. Moreover, for $i=1,2$, functions $a_i(x)$ are true in the following condition:

(A_0)

$$a_i \in L^\infty(\Omega), \quad a_i^0 := ess\inf_{x\in\Omega} a_i(x) > 0.$$

If $T:\Omega\times\mathbb{R}^2\to\mathbb{R}$ then, we suppose following assumption on T:

(T_0) $T:\Omega\times\mathbb{R}^2\to\mathbb{R}$ is measurable in Ω for all $(s,t)\in\mathbb{R}^2$ and $T(x,.,.)$ is C^1 with respect to $(s,t)\in\mathbb{R}^2$ for a.e. $x\in\Omega$ and for each $\theta>0$,

$$\sup_{|(s,t)|\leq\theta}|T_u(.,s,t)|, \sup_{|(s,t)|\leq\theta}|T_v(.,s,t)| \in L^1(\Omega).$$

2. Preliminaries and notations

We start by introducing the anisotropic variable exponent Sobolev spaces. We consider the vectorial function $\overrightarrow{p}:\overline{\Omega}\to\mathbb{R}^N$ with $\overrightarrow{p}(x)=(p_1(x),\cdots,p_N(x))$ that $p_i\in C_+(\overline{\Omega})$ for all $i\in\{1,\cdots,N\}$. We set

$$p^- := \inf_{x\in\Omega} p(x), \qquad p^+ := \sup_{x\in\Omega} p(x),$$

$$\underline{p}=\min\left\{p_i^- : \quad i=1,\cdots,N\right\}, \quad \overline{p}=\max\left\{p_i^+ : \quad i=1,\cdots,N\right\}.$$

The anisotropic variable exponent Sobolev space is defined as follows

$$W^{1,\overrightarrow{p}(x)}(\Omega)=\left\{u\in L^{p_i(x)}(\Omega): \frac{\partial u}{\partial x_i}\in L^{p_i(x)}(\Omega) \text{ for } i=1,\cdots,N\right\},$$

with the norm $\|u\|_{\overrightarrow{p}} := \|u\|_{W^{1,\overrightarrow{p}(x)}(\Omega)} = \sum_{i=1}^N\left(\|\frac{\partial u}{\partial x_i}\|_{p_i}+\|u\|_{p_i}\right)$. The space $(W^{1,\overrightarrow{p}(x)}(\Omega),\|\cdot\|_{\overrightarrow{p}})$ is a separable and reflexive Banach space. We consider the product space

$$X := W^{1,\overrightarrow{p}(x)}(\Omega)\times W^{1,\overrightarrow{p}(x)}(\Omega)$$

which is equipped with the norm $\|(u,v)\| := \|u\|_{\overrightarrow{p}} + \|v\|_{\overrightarrow{p}}$. Define the functionals $\Phi, \Psi_{\lambda,\mu} : X \to \mathbb{R}$, by

$$\Phi(u,v) := \sum_{i=1}^{N} \left(\int_\Omega \frac{1}{p_i(x)} |\frac{\partial u}{\partial x_i}|^{p_i(x)} dx + \int_\Omega \frac{a_1(x)}{p_i(x)} |u|^{p_i(x)} dx \right)$$
$$+ \sum_{i=1}^{N} \left(\int_\Omega \frac{1}{p_i(x)} |\frac{\partial v}{\partial x_i}|^{p_i(x)} dx + \int_\Omega \frac{a_2(x)}{p_i(x)} |v|^{p_i(x)} dx \right), \tag{2.1}$$

and

$$\Psi_{\lambda,\mu}(u,v) := \int_\Omega F(x,u,v)dx + \frac{\mu}{\lambda} \int_\Omega G(x,u,v)dx, \tag{2.2}$$

for any $(u,v) \in X$. set $I_{\lambda,\mu} = \Phi(u,v) - \lambda \Psi_{\lambda,\mu}(u,v)$. To prove the main theorem, we need the following lemma which we have proved in this article.

Lemma 2.1. *set* $U(x) = \sum_{i=1}^{N} \left(\int_\Omega |\frac{\partial u}{\partial x_i}|^{p_i(x)} dx + \int_\Omega a(x)|u|^{p_i(x)} dx \right)$ *for all* $u \in W^{1,\overrightarrow{p}(x)}(\Omega)$. *So, there exist constants* $\beta_1, \beta_2 > 0$ *that*

(i) $\|u\|_{\overrightarrow{p}} \geq 1 \Longrightarrow \beta_1 \|u\|_{\overrightarrow{p}}^{\underline{p}} \leq U(x) \leq \beta_2 \|u\|_{\overrightarrow{p}}^{\overline{p}}$,

(ii) $\|u\|_{\overrightarrow{p}} \leq 1 \Longrightarrow \beta_1 \|u\|_{\overrightarrow{p}}^{\overline{p}} \leq U(x) \leq \beta_2 \|u\|_{\overrightarrow{p}}^{\underline{p}}$.

3. MAIN RESULT

In the following, we will state the main theorem.

Theorem 3.1. *Suppose that*

(A_1) *for each* $(x,s,t) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^+$, $F(x,s,t) \geq 0$;

(A_2) *there exist* $\alpha \in L^\infty(\Omega), \alpha(x) > 0$ *a.e. in* Ω *and* $\gamma_1, \gamma_2 \in C_+$ *with* $0 < \gamma_1(x) \leq \gamma_1^+ < \gamma_2^+ < \frac{\underline{p}}{2}$ *such that*

$$|F(x,s,t)|, |G(x,s,t)| \leq \alpha(x) \left(1 + |s|^{\gamma_1(x)} + |t|^{\gamma_2(x)}\right)$$

for a.e. $x \in \Omega$ *and each* $(s,t) \in \mathbb{R}^2$;

(A_3) *there exist two positive constants* δ *and* τ *such that*

$$c_0^{\underline{p}} C_{\underline{p}}^2 N^{\underline{p}} (a_1^0 + a_2^0) meas(\Omega) \min\{\delta^{\underline{p}}, \delta^{\overline{p}}\} > \min\{1, a_1^0, a_2^0\} \tau^{\underline{p}};$$

(A_4)

$$\frac{\int_\Omega \sup_{|(s,t)| \leq \tau} F(x,s,t)dx}{\tau^{\underline{p}}} < \frac{\underline{p} \min\{1, a_1^0, a_2^0\} \int_\Omega F(x,\delta,\delta)dx}{\overline{p} C_0^{\underline{p}} C_{\underline{p}}^2 N^{\underline{p}} (\|a_1\|_\infty + \|a_2\|_\infty) meas(\Omega) \max\{\delta^{\overline{p}}, \delta^{\underline{p}}\}};$$

so, for each $\lambda \in \Lambda_{\delta,\tau}$, *given by*

$$\left] \frac{(\|a_1\|_\infty + \|a_2\|_\infty) N meas(\Omega) \max\{\delta^{\overline{p}}, \delta^{\underline{p}}\}}{\underline{p} \int_\Omega F(x,\delta,\delta)dx}, \frac{\min\{1, a_1^0, a_2^0\} \tau^{\underline{p}}}{\overline{p} C_0^{\underline{p}} C_{\underline{p}}^2 N^{\underline{p}-1} \int_\Omega \sup_{|(s,t)| \leq \tau} F(x,s,t)dx} \right[, \tag{3.1}$$

and for every $G : \Omega \times \mathbb{R}^2 \to \mathbb{R}$, there is $\varepsilon > 0$ given by $\varepsilon = \min\{\mathcal{A}_\tau, \mathcal{B}_\delta\}$, where

$$\mathcal{A}_\tau = \frac{\min\{1, a_1^0, a_2^0\}\tau^{\underline{p}} - \lambda\overline{p}C_0^{\underline{p}}C_{\underline{p}}^2 N^{\underline{p}-1}\int_\Omega \sup_{|(s,t)|\leq\tau} F(x,s,t)dx}{\overline{p}C_0^{\underline{p}}C_{\underline{p}}^2 N^{\underline{p}-1}\int_\Omega sup_{|(s,t)|\leq\tau} G(x,s,t)dx},$$

$$\mathcal{B}_\delta = \frac{\lambda\underline{p}\int_\Omega F(x,\delta,\delta)dx - N(\|a_1\|_\infty + \|a_2\|_\infty)meas(\Omega)\max\left\{\delta^{\overline{p}}, \delta^{\underline{p}}\right\}}{\underline{p}\int_\Omega G(x,\delta,\delta)dx},$$

*such that for each $\mu \in [0, \varepsilon[$, the problem (**??**) admits at least three distinct weak solutions.*

Proof. Using the critical points theorem of Bonanno and Marano[**?**], we prove the existence at least three distinct weak solutions for system (**??**). we showed that Φ is coercive and functions Φ and Ψ hold in the conditions of the three critical points theorem of Bonanno, that's mean,

- $\Phi, \Psi_{\lambda,\mu} \in C^1(X, \mathbb{R})$ [**?**, Lemma 3.4].
- The functional Φ is sequentially weakly lower semicontinuous.
- $\Psi'_{\lambda,\mu} : X \to X^*$ is a compact operator.
- Φ' admits a continuous inverse on X^*.

In the following, for $\delta > 0$, we pick $w(x) := (\delta, \delta)$ for any $x \in \Omega$ and

$$r := \frac{\min\left\{1, a_1^0, a_2^0\right\}}{\overline{p}C_{\underline{p}}^2 N^{\underline{p}-1}}\left(\frac{\tau}{C_0}\right)^{\underline{p}}.$$

We show that for $\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)}\Psi(u,v)}\right[$,
the functional $I_{\lambda,\mu}$ is coercive. Therefore, all the conditions of Bonanno's theorem are satisfied and we can conclude that the functional $I_{\lambda,\mu}$ admits at least three critical points in X which are the weak solutions of system (**??**). $\square$

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Oral Presentation $\ast$: **Speaker**

TERNARY FUZZY DERIVATIONS ON TERNARY FUZZY BANACH ALGEBRAS

BAHMAN TAHERKHANI*AND HASAN NARAGHI

Department of Mathematics, Payame Noor University, Tehran, Iran.
Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Arak, Iran.
bahman.taherkhani@pnu.ac.ir and hassannarghi1976@gmail.com

Abstract. In this paper, we introduce the concept of the generalized $\mathbb{C}$-linear s-functional inequalities and we investigate Hyers-Ulam stability by using the fixed point method of ternary fuzzy derivation and ternary Jordan fuzzy derivation for the $\mathbb{C}$-linear s-functional inequalities on ternary fuzzy Banach algebra.

1. Introduction

The stability problem of functional equations originated from the question of Ulam in 1940 concerning the stability of group homomorphisms. Hyers in 1941 gave the first affirmative partial answer to the question of Ulam for Banach spaces. Th. M. Rassias in 1978 proved the Hyers-Ulam stability by changing of Hyers' theorem control function with $\theta(\|x\|^P + \|y\|^p)$ for linear mappings and J. M. Rassias in 1982, investigated the Hyers-Ulam-Rassias stability by replacing of $\theta(\|x\|^P + \|y\|^p)$ with $\theta(\|x\|^P.\|y\|^p)$. Finally, Găvruta in 1994, generalized the Rassias' result.

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* Sspeaker.

In the follwing, we use the definition of fuzzy normed spaces to investigate a fuzzy version of the Hyers-Ulam stability for the functional equation in the fuzzy normed algebra.

Definition 1.1. [1, 2] Let X be a vector space. A function $f : X\times\mathbb{R} \to [0, 1]$ is called a fuzzy norm on X if (1): $N(x, t) = 0$ for all $x \in X$ and $t \in \mathbb{R}$ with $t \leq 0$;
(2): $x = 0$ if and only if $N(x, t) = 1$ for all $x \in X$ and $t \geq 0$;
(3): $N(cx, t) = N(x, \frac{t}{|c|})$ for all $x \in X$ and $c \neq 0$;
(4): $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$ for all $x, y \in X$ and $t, s \in \mathbb{R}$;
(5): N(x,.) is a decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$ for all $x \in X$ and $t \in \mathbb{R}$;

Definition 1.2. [1, 2] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ or converges if there exists $x \in X$ such that

$$\lim_{t\to\infty} N(x_n - x, t) = 1$$

for all $t > 0$ In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{t\to\infty} x_n = x$.

Definition 1.3. [1, 2] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if, for each $\varepsilon > 0$ and $t > 0$ there exists an $n_0 \in N$ such that for all $n \geq n_0$ and all $p > 0$ we have is said to be convergent to a point $x \in X$ or converges if there exists $x \in X$ such that

$$N(x_{n+p} - x_n, t) > 1 - \varepsilon.$$

It is well known that every convergent sequence in a fuzzy normed vector space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if, for each sequence $\{X_n\}$ converging to $x_0 \in X$, the sequence $f(x_n)$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X. Consider the generalized $\mathbb{C}$-linear s-functional equation

$$\begin{aligned} &\Big\| f\Big(\lambda(x+y+z)\Big) - \lambda f(x) - \lambda f(y) - \lambda f(z)\Big\| \leq \\ &\Big\| s\Big(f\Big(\lambda(x+y+z)\Big) + \lambda f(x) - f\Big(\lambda(x+y)\Big) - f\Big(\lambda(y+z)\Big)\Big)\Big\| \end{aligned} \tag{1.1}$$

where s is a fixed nonzero complex number with $|s| < 1$.

Definition 1.4. [2, 3] Let (X, N) and (Y, N') be two ternary fuzzy normed algebras. A $\mathbb{C}$-linear mapping $D : (X, N) \to (X, N)$ is called a ternary fuzzy

derivation if
(1) ternary derivation if

$$D\Big([x,y,z]\Big) = [D(x),y,z] + [x,D(y),z] + [x,y,D(z)]$$

(2) ternary Jordan derivation if

$$D\Big([x,x,x]\Big) = [D(x),x,x] + [x,D(x),x] + [x,x,D(x)]$$

for all $x, y, z \in \mathfrak{A}$.

In this paper, we solve (1.1) and show that a function which satisfies (1.1) is $\mathbb{C}$-linear. We also prove its Hyers-Ulam stability by using the fixed point method [4] of ternary fuzzy derivations on ternary fuzzy Banach algebras.

2. Main Results

Throughout this section, assume that X be a ternary fuzzy Banach algebra. For any mapping $f : X \to Y$, we define

$$\begin{aligned}\Delta_\lambda(x,y,z) := \Big\|f\Big(\lambda(x+y+z)\Big) - \lambda f(x) - \lambda f(y) - \lambda f(z)\Big\| \leq \\ \Big\|s\Big(f\Big(\lambda(x+y+z)\Big) + \lambda f(x) - f\Big(\lambda(x+y)\Big) - f\Big(\lambda(y+z)\Big)\Big)\Big\|\end{aligned}$$

and

$$d(x,y,z) := f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]$$

for all $x, y, z \in X$
We need following lemma to prove the main theorems.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies*

$$\begin{aligned}&f\Big(\lambda(x+y+z)\Big) - \lambda f(x) - \lambda f(y) - \lambda f(z) = \\ &\rho\Big(f\Big(\lambda(x+y+z)\Big) + \lambda f(x) - f\Big(\lambda(x+y)\Big) - f\Big(\lambda(y+z)\Big)\Big)\end{aligned} \tag{2.1}$$

for all $x, y, z \in X$, then the mapping f is additive.

In the following theorem, we prove Hyers-Ulam stability of ternary fuzzy derivation.

Theorem 2.2. *Let $f : X \to Y$ be a mapping for which there exist functions $\sigma : X^3 \to [0,\infty)$ such that there exists an $0 < L < 1$ with*

$$\sigma(\frac{x}{2},\frac{y}{2},\frac{z}{2}) \leq \frac{L}{2}\sigma(x,y,z) \tag{2.2}$$

$$N_X\Big(\Delta_\lambda(x,y,z),t\Big) \geq \frac{t}{t+\sigma(x,y,z)} \tag{2.3}$$

(i) : Suppose that

$$N_X\Big(d[x,y,z],t\Big) \geq \frac{t}{t+\sigma(x,y,z)} \tag{2.4}$$

for all $x, y, z \in X$. Then there exists a unique ternary fuzzy derivation $D : X \to X$ such that

$$N_X\Big(f(x)-D(x),t\Big) \geq \frac{(6-6L)t}{(6-6L)t+\sigma(x,x,0)} \tag{2.5}$$

for all $x \in X$.
(ii) : Suppose that

$$N_X\Big(d[x,x,x],t\Big) \geq \frac{t}{t+\sigma(x,x,x)} \tag{2.6}$$

for all $x \in X$. Then there exists a unique ternary Jordan fuzzy derivation $D : X \to X$ such that

$$N_X\Big(f(x)-D(x),t\Big) \geq \frac{(6-6L)t}{(6-6L)t+\sigma(x,x,0)} \tag{2.7}$$

for all $x \in X$.

In the following theorem, we prove Hyers-Ulam-Rassias stability of ternary fuzzy derivation and ternary Jordan fuzzy derivation with condition $0 < r < 1$.

Theorem 2.3. *Let $\theta > 0$ and $f : X \to X$ be a mapping such that*

$$N_X\Big(\Delta_\lambda(x,y,z),t\Big) \geq \frac{t}{t+\theta(\|x\|^r+\|y\|^r+\|z\|^r)} \tag{2.8}$$

(i) : Suppose that

$$N_X\Big(d[x,y,z],t\Big) \geq \frac{t}{t+\theta(\|x\|^r\|y\|^r\|z\|^r)} \tag{2.9}$$

for all $x, y, z \in X$. Then there exists a unique ternary fuzzy derivation $D : X \to X$ such that

$$N_X\Big(f(x)-D(x),t\Big) \geq \frac{t}{t+\theta(\|x\|^r)} \tag{2.10}$$

for all $x \in X$.
(ii) : Suppose that

$$N_X\Big(d[x,x,x],t\Big) \geq \frac{t}{t+\theta(\|x\|^r\|y\|^r\|z\|^r)} \tag{2.11}$$

for all $x \in X$. Then there exists a unique ternary Jordan fuzzy derivation $D : X \to X$ such that

$$N_X\Big(f(x)-D(x),t\Big) \geq \frac{t}{t+\theta(\|x\|^r)} \tag{2.12}$$

for all $x \in X$.

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Oral Presentation

CHARACTERIZATION OF n-JORDAN MULTIPLIERS THROUGH ZERO PRODUCTS

ABBAS ZIVARI-KAZEMPOUR

Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran
zivari@abru.ac.ir, zivari6526@gmail.com

Abstract. Let A be a unital C^*-algebra and X be a unital Banach A-bimodule. We characterize n-Jordan multipliers $T : A \longrightarrow X$ through the action on zero product. We also prove that each continuous linear mapping T from group algebra $L^1(G)$ into unital Banach A-bimodule X which satisfies a related condition is an n-Jordan multiplier.

1. Introduction

Let A be a Banach algebra and X be an A-bimodule. A linear map $T : A \longrightarrow X$ is called *left multiplier* [*right multiplier*] if for all $a, b \in A$,

$$T(ab) = T(a)b, \quad [T(ab) = aT(b)],$$

and T is called a *multiplier* if it is both left and right multiplier. Also, T is called *left Jordan multiplier* [*right Jordan multiplier*] if for all $a \in A$,

$$T(a^2) = T(a)a, \quad [T(a^2) = aT(a)],$$

and T is called a *Jordan multiplier* if T is a left and a right Jordan multiplier.

It is clear that every left (right) multiplier is a left (right) Jordan multiplier, but the converse is not true in general, see for example [6].

A linear map D from Banach algebra A into an A-bimodule X is called *derivation* [*Jordan derivation*] if

$$D(ab) = D(a)b + aD(b), \quad [D(a^2) = D(a)a + aD(a)], \quad a, b \in A.$$

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Note that every derivation is a Jordan derivation, but the converse is fails in general [5]. It is proved by B. E. Johnson in [5, Theorem 6.3] that every Jordan derivation from C^*-algebra A into any A-bimodule X is a derivation.

Definition 1.1. Let A be a Banach algebra, X be a right A-module and let $T : A \longrightarrow X$ be a linear map. Then T is called *left n-Jordan multiplier* if for all $a \in A$, $T(a^n) = T(a^{n-1})a$. The *right n-Jordan multiplier* and *n-Jordan multiplier* can be defined analogously.

The two following results concerning characterization of n-Jordan multiplier presented by the first author in [6].

Theorem 1.2. [6, Theorem 2.3] *Let A be a unital Banach algebra and X be a unital Banach left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map such that*

$$a, b \in A, \quad ab = e_A \quad \Longrightarrow \quad T(ab) = aT(b). \tag{1.1}$$

Then T is a right n-Jordan multiplier.

Lemma 1.3. [6, Lemma 2.1] *Let A be a Banach algebra, X be a left A-module and let $T : A \longrightarrow X$ be a right Jordan multiplier. Then T is a right n-Jordan multiplier for each $n \geq 2$.*

Let A be a Banach algebra and X be a arbitrary Banach space. Then the continuous bilinear mapping $\phi : A \times A \longrightarrow X$ preserves zero products if

$$ab = 0 \quad \Longrightarrow \quad \phi(a, b) = 0, \quad a, b \in A. \tag{1.2}$$

Motivated by (1.2) the following concept was introduced in [1].

Definition 1.4. A Banach algebra A *has the property* $(\mathbb{B})$ if for every continuous bilinear mapping $\phi : A \times A \longrightarrow X$, where X is an arbitrary Banach space, the condition (1.2) implies that $\phi(ab, c) = \phi(a, bc)$, for all $a, b, c \in A$.

It is known that every C^*-algebra A and the group algebra $L^1(G)$ for a locally compact group G has the property $(\mathbb{B})$, [1].

Let $\mathfrak{J}(A)$ denote the subalgebra of A generated by all idempotents in A. If $A = \overline{\mathfrak{J}(A)}$, then we say that the Banach algebra A is generated by idempotents. Examples of such Banach algebras are given in [1].

Consider the following condition on a linear map T from Banach algebra A into a Banach A-bimodule X which is related to the condition (1.1).

$$a, b \in A, \quad ab = 0 \quad \Longrightarrow \quad aT(b) = 0. \tag{1.3}$$

A rather natural weakening of condition (1.3) is the following:

$$a, b \in A, \quad ab = ba = 0 \quad \Longrightarrow \quad aT(b) + bT(a) = 0. \tag{1.4}$$

In this note, according to [7], we investigate whether those conditions characterizes n-Jordan multipliers.

2. Characterization of n-Jordan multipliers

Since all results which are true for left versions have obvious analogue statements for right versions, we will focus in the sequel just the right versions.

Theorem 2.1. *Let A be a unital C^*-algebra and X be a unital left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map satisfying (1.3). Then T is a right n-Jordan multiplier.*

We mention that Theorem 2.1 is also true for non-unital case, because every C^*-algebra A has a bounded approximate identity.

In view of Theorem 2.1, the next question can be raised. Dose Theorem 2.1 remain valid with condition (1.3) replaced by (1.4)?

Theorem 2.2. [2, Theorem 2.2] *Let A be a C^*-algebra and X be a Banach space and let $\phi : A \times A \longrightarrow X$ be a continuous bilinear mapping such that*

$$ab = ba = 0 \quad \Longrightarrow \quad \phi(a, b) = 0, \qquad a, b \in A.$$

Then

$$\phi(ax, by) + \phi(ya, xb) = \phi(a, xby) + \phi(yax, b),$$

for all $a, b, x, y \in A$.

Our first main result is the following.

Theorem 2.3. *Let A be a unital C^*-algebra and X be a symmetric unital left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map satisfying (1.4). Then there exist a Jordan derivation D and a Jordan multiplier ψ such that $T = D + \psi$.*

Corollary 2.4. *Let A be a commutative unital C^*-algebra and X be a symmetric unital left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear mapping such that the condition (1.4) holds. Then T is an n-Jordan multiplier.*

Next we generalize Corollary 2.4 and give the affirmative answer to the preceding question.

Theorem 2.5. *Let A be a von Neumann algebra and X be a unital left A-module. If $T : A \longrightarrow X$ is a continuous linear map satisfying (1.4), then T is a right n-Jordan multiplier.*

It is shown [3] that every C^*-algebra A is Arens regular and the second dual of each C^*-algebra is a von Neumann algebra. Hence by extending the continuous linear map $T : A \longrightarrow X$ to the second adjoint $T^{**} : A^{**} \longrightarrow X^{**}$ and applying Theorem 2.5, we get the following result.

Corollary 2.6. *Suppose that A is a unital C^*-algebra and X is a unital left A-module. If $T : A \longrightarrow X$ is a continuous linear map satisfying (1.4), then T is a right n-Jordan multiplier.*

Theorem 2.7. [4, Corollary 3.6] *Let A be Banach algebra, X be a Banach space and $\phi : A \times A \longrightarrow X$ be a continuous bilinear mapping such that*

$$a, b \in A, \quad ab = ba = 0 \implies \phi(a, b) = 0,$$

then

$$\phi(a, x) + \phi(x, a) = \phi(ax, e_A) + \phi(e_A, xa),$$

for all $a \in A$ and $x \in \mathfrak{J}(A)$. In particular, if A is generated by idempotents, then

$$\phi(a, b) + \phi(b, a) = \phi(ab, e_A) + \phi(e_A, ba), \qquad a, b \in A.$$

By using Theorem 2.7 we can obtain the following result.

Theorem 2.8. *Let A be a unital Banach algebra which is generated by idempotents and X be a symmetric unital left A-module. If $T : A \longrightarrow X$ is a continuous linear map satisfying (1.4), then there exist a Jordan derivation D and a Jordan multiplier ψ such that $T = D + \psi$.*

Corollary 2.9. *Let A be a commutative unital Banach algebra such that $A = \overline{\mathfrak{J}(A)}$ and X be a symmetric unital left A-module. Let $T : A \longrightarrow X$ be a continuous linear map satisfying (1.4). Then T is an n-Jordan multiplier.*

Let $A = L^1(G)$ for a locally compact abelian group G. Then A is commutative and it is weakly amenable [3], but neither it is C^*-algebra nor generated by idempotents. Therefore Corollary 2.4 and Corollary 2.9 cannot be applied for it.

The following result shows that analogous of Corollary 2.4 is also true for group algebra.

Theorem 2.10. *Let $A = L^1(G)$ for a locally compact abelian group G. Suppose that X is a symmetric unital left A-module and $T : A \longrightarrow X$ is a continuous linear map satisfying (1.4). Then T is an n-Jordan multiplier.*

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Posters

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Poster Presentation

ON ORTHOGONALLY PEXIDER FUNCTIONAL EQUATION $f(x+y)+g(x-y)=h(x)+k(y)$

SAYED KHALIL EKRAMI

Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran.
khalil.ekrami@gmail.com, ekrami@pnu.ac.ir

Abstract. One of the pexiderized types of the orthogonally quadratic functional equation is of the form

$$f(x+y)+g(x-y)=h(x)+k(y) \quad (x \perp y).$$

In this paper, we investigate the general solution of this orthogonally pexider functional equation on an orthogonality space in the sense of Rätz, where the function g is odd.

1. Introduction

J. Rätz introduced a defnition of an abstract orthogonality by using four axioms on a real vector space X with $\dim X \geq 2$ (See [3]). Suppose X is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0$ and $0 \perp x$ for all $x \in X$;
(O_2) independence: if $x, y \in X - \{0\}$ and $x \perp y$, then x, y are linearly independent;
(O_3) homogeneity: if $x, y \in X, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
(O_4) the Thalesian property: if P is a 2-dimensional subspace of X, for any $x \in P$ and any $\lambda \in \mathbb{R}^+$, there exists $y \in P$ such that $x \perp y$ and $x+y \perp \lambda x - y$.

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The pair $(X, \perp)$ is called an orthogonality space. Some interesting examples of orthogonality spaces are

(a) Any real vector space X can be made into a orthogonality space with the trivial orthogonality defined on X by (i) for all $x \in X$, $x \perp 0$ and $0 \perp x$, (ii) for all $x, y \in X \setminus \{0\}$, $x \perp y$ if and only if x, y are linearly independent.

(b) Any inner product space $(X, \langle \cdot, \cdot \rangle)$ is an orthogonality space with the ordinary orthogonality given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(c) Any normed space $(X, \| \cdot \|)$ can be made into a orthogonality space with the Birkhoff-James orthogonality defined by $x \perp y$ if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly examples (a) and (b) are symmetric but example (c) is not.

Let H be an inner product space with $\dim H > 2$ with the usual orthogonality given by $x \perp y \Leftrightarrow \langle x, y \rangle = 0$. Suppose that the functions $f, g, h, k : H \to \mathbb{R}$ satisfy the orthgonally pexider functional equation $f(x + y) + g(x - y) = h(x) + k(y)$ $(x \perp y)$ $\quad (*)$. Fochi [1] showed that the general solution of $(*)$ is of the form

$$\begin{aligned}
f(x) &= \tfrac{1}{2}\big(Q(x) + A(x) + B(x) + \phi(\|x\|) + h(0) + k(0)\big),\\
g(x) &= \tfrac{1}{2}\big(Q(x) + A(x) - B(x) - \phi(\|x\|) + h(0) + k(0)\big),\\
h(x) &= Q(x) + A(x) + h(0),\ k(x) = Q(x) + B(x) + k(0),
\end{aligned}$$

where $Q : H \to \mathbb{R}$ is a quadratic function, $A, B : H \to \mathbb{R}$ are additive functions and $\phi : [0, \infty) \to \mathbb{R}$ defined by $\phi(\|x\|) = f^e(x) - g^e(x)$ in which f^e and g^e are the even part of f and the even part of g, respectively.

In this paper, let $(X, \perp)$ be an orthogonality space in which $\perp$ is symmetric and Y be a real vector space. We investigate the general solution of $(*)$, where the function g is odd.

2. **The Result**

In this section, we investigate the general solution of $(*)$, where the orthogonality is in the sense of Rätz and the function g is odd.

Lemma 2.1. *Let $(X, \perp)$ be an orthogonality space and Y be a vector space. If the odd function $A : X \to Y$ satisfies the orthgonally functional equation $A(x + y) + A(x - y) = 2A(x)$ $\quad (x \perp y)$, then A is additive.*

Proof. Let $x, y \in X$ with $x \perp y$. Interchanging x with y in $A(x+y)+A(x-y) = 2A(x)$, we get $A(x + y) - A(x - y) = 2A(y)$. By these equations we have $A(x + y) = A(x) + A(y)$. Thus A is orthogonally additive and since A is odd, so on account of Theorem 5 of [3], it is additive. $\square$

Theorem 2.2. *Let $(X, \perp)$ be an orthogonality space, where $\perp$ is symmetric and Y be a vector space. If the functions $f, g, h, k : X \to Y$ satisfy the* *orthgonally pexider functional equation*

$$x \perp y \quad \Rightarrow \quad f(x+y)+g(x-y)=h(x)+k(y), \tag{2.1}$$

and the function g is odd, then there exist orthogonally quadratic function $Q : X \to Y$ and additive functions $A, B : X \to Y$ such that

$$f(x)=Q(x)+\tfrac{1}{2}\big(A(x)+B(x)\big)+f(0), \ \ g(x)=\tfrac{1}{2}\big(A(x)-B(x)\big),$$
$$h(x)=Q(x)+A(x)+h(0), \ \ k(x)=Q(x)+B(x)+k(0).$$

Proof. Putting $x=y=0$ in (2.1), we get

$$f(0)=h(0)+k(0). \tag{2.2}$$

Also putting $y=0$ and $x=0$ respectively in (2.1), we get

$$f(x)+g(x)=h(x)+k(0), \tag{2.3}$$

$$f(y)+g(-y)=h(0)+k(y), \tag{2.4}$$

for all $x, y \in X$. Replacing y by $-x$ in (2.4), we have

$$f(-x)+g(x)=h(0)+k(-x). \tag{2.5}$$

From (2.3) and (2.5), we have $f(x)-f(-x)=h(x)-k(-x)+k(0)-h(0)$. Replacing x by $-x$ in the last equation, we get $f(-x)-f(x)=h(-x)-k(x)+k(0)-h(0)$. Using the last two equations, we obtain $h(x)+h(-x)-2h(0)=k(x)+k(-x)-2k(0) \quad (x \in X)$. Define

$$Q(x) := \tfrac{1}{2}\big(h(x)+h(-x)\big)-h(0)=\tfrac{1}{2}\big(k(x)+k(-x)\big)-k(0) \quad (x \in X), \tag{2.6}$$

then Q is an even function and $Q(0)=0$.

Replacing x by $-x$ in (2.1), we get

$$f(-x+y)+g(-x-y)=h(-x)+k(y) \quad (x \perp y). \tag{2.7}$$

From (2.1) and (2.7) (adding and subtracting, respectively), we get

$$f(x+y)+g(x-y)+f(-x+y)+g(-x-y)=2Q(x)+2k(y)+2h(0) \quad (x \perp y), \tag{2.8}$$

$$f(x+y)+g(x-y)-f(-x+y)-g(-x-y)=h(x)-h(-x) \quad (x \perp y). \tag{2.9}$$

Define the function $A : X \to Y$ by $A(x) := \frac{1}{2}\big(h(x)-h(-x)\big)$ $(x \in X)$, then A is an odd function and so $A(0)=0$. Using (2.8) and (2.9), we obtain

$$f(x+y)+g(x-y)=Q(x)+A(x)+h(0)+k(y) \quad (x \perp y),$$

and then by (2.1), we have

$$h(x)=Q(x)+A(x)+h(0) \quad (x \in X). \tag{2.10}$$

Replacing y by $-y$ in (2.1), we get

$$f(x-y)+g(x+y)=h(x)+k(-y) \quad (x \perp y). \tag{2.11}$$

From (2.1) and (2.11) (adding and subtracting, respectively), we get

$$f(x+y)+g(x-y)+f(x-y)+g(x+y)=2h(x)+2Q(y)+2k(0) \quad (x \perp y), \tag{2.12}$$

$$f(x+y)+g(x-y)-f(x-y)-g(x+y)=k(y)-k(-y) \quad (x \perp y). \tag{2.13}$$

Define the function $B : X \to Y$ by $B(x) := \frac{1}{2}\big(k(x) - k(-x)\big)$ $(x \in X)$, then B is an odd function and so $B(0) = 0$. Using (2.12) and (2.13), we obtain

$$f(x+y) + g(x-y) = Q(y) + B(y) + h(x) + k(0) \quad (x \perp y),$$

and then by (2.1), we have

$$k(y) = Q(y) + B(y) + k(0) \quad (y \in X). \tag{2.14}$$

Using (2.1), (2.10) and (2.14), we get

$$f(x+y)+g(x-y) = Q(x)+Q(y)+A(x)+B(y)+h(0)+k(0) \quad (x \perp y). \tag{2.15}$$

Putting $y = 0$ and $x = 0$ respectively in (2.15) and using (2.2), we get

$$f(x) + g(x) = Q(x) + A(x) + f(0) \quad (x \in X),$$
$$f(x) - g(x) = Q(x) + B(x) + f(0) \quad (x \in X).$$

From the last equations, we obtain

$$f(x) = Q(x) + \tfrac{1}{2}\big(A(x) + B(x)\big) + f(0) \quad (x \in X), \tag{2.16}$$

$$g(x) = \tfrac{1}{2}\big(A(x) - B(x)\big) \quad (x \in X). \tag{2.17}$$

It remains to show that Q is orthogonality quadratic and A, B are additive. From (2.16), we have $f(-x) = Q(x) - \frac{1}{2}\big(A(x)+B(x)\big) + f(0)$, and so $f(x) + f(-x) = 2Q(x) + 2f(0)$ which implies that

$$Q(x) = \tfrac{1}{2}\big(f(x) + f(-x)\big) - f(0) \quad (x \in X). \tag{2.18}$$

Interchanging x by y in (2.15), we have

$$f(x+y) - g(x-y) = Q(x) + Q(y) + A(y) + B(x) + f(0) \quad (x \perp y).$$

Using the last equation and (2.15), we obtain

$$2f(x+y) = 2Q(x) + 2Q(y) + A(x) + A(y) + B(x) + B(y) + 2f(0) \quad (x \perp y),$$

which implies that

$$f(x+y) = Q(x) + Q(y) + \tfrac{1}{2}\big(A(x) + A(y) + B(x) + B(y)\big) + f(0) \quad (x \perp y). \tag{2.19}$$

Let $x, y \in X$ with $x \perp y$, using (2.18) and (2.19), we can conclude that

$$\begin{aligned}
&Q(x+y) + Q(x-y)\\
&= \tfrac{1}{2}\Big(f(x+y) + f(-x-y)\Big) - f(0) + \tfrac{1}{2}\Big(f(x-y) + f(-x+y)\Big) - f(0)\\
&= \tfrac{1}{2}\Big(f(x+y) + f(-x-y) + f(x-y) + f(-x+y)\Big) - 2f(0)\\
&= \tfrac{1}{2}\Big(Q(x) + Q(y) + \tfrac{1}{2}\big(A(x) + A(y) + B(x) + B(y)\big) + f(0)\\
&\quad + Q(x) + Q(y) + \tfrac{1}{2}\big(-A(x) - A(y) - B(x) - B(y)\big) + f(0)\\
&\quad + Q(x) + Q(y) + \tfrac{1}{2}\big(A(x) - A(y) + B(x) - B(y)\big) + f(0)\\
&\quad + Q(x) + Q(y) + \tfrac{1}{2}\big(-A(x) + A(y) - B(x) + B(y)\big) + f(0)\Big) - 2f(0)\\
&\quad = 2Q(x) + 2Q(y).
\end{aligned}$$

Thus the function Q is orthogonally quadratic. From (2.16) and (2.17), we get

$$A(x)=f(x)+g(x)-Q(x)-f(0) \quad (x \in X). \tag{2.20}$$

Thus for any $x, y \in X$ with $x \perp y$, by (2.15) and (2.20), we get

$$\begin{aligned}
&A(x+y)+A(x-y)\\
&=f(x+y)+g(x+y)-Q(x+y)-f(0)\\
&\quad+f(x-y)+g(x-y)-Q(x-y)-f(0)\\
&=Q(x)+Q(y)+A(x)+B(y)+f(0)+Q(x)+Q(y)+A(x)-B(y)+f(0)\\
&\quad-Q(x+y)-Q(x-y)-2f(0)=2A(x).
\end{aligned}$$

Hence by Lemma 2.1, A is additive. This completes the proof. □

3. Conclusion

Let $(X, \perp)$ be an orthogonality space in which $\perp$ is symmetric and Y be a real vector space. In this paper, we investigate the general solution of the orthgonally pexider functional equation $f(x+y)+g(x-y)=h(x)+k(y)$ $(x \perp y)$, where the function g is odd.

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ON ORTHOGONALLY ADDITIVE ISOMETRIES

SAYED KHALIL EKRAMI

Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran.
khalil.ekrami@gmail.com, ekrami@pnu.ac.ir

Abstract. Let H be a real inner product space. In this paper, we show that if a mapping $f : H \to H$ satisfies

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in H$ with $x \perp y$ and

$$\|f(x)\| = \|x\|$$

for all $x, \in H$, then f is an additive mapping.

1. Introduction

There are several orthogonality notions on a real normed space such as Birkhoff-James, isosceles, Phythagorean, Roberts and Diminnie ([3]). J. Rätz [1] introduced an abstract definition of orthogonality on a real vector space by using four axioms. Let us recall the orthogonality in the sense of Rätz.

Definition 1.1. Suppose X is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0$ and $0 \perp x$ for all $x \in X$;
(O_2) independence: if $x, y \in X \setminus \{0\}$ and $x \perp y$, then x, y are linearly independent;
(O_3) homogeneity: if $x, y \in X, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

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(O_4) the Thalesian property: if P is a 2-dimensional subspace of X, for any $x \in P$ and any $\lambda \in \mathbb{R}^+$, there exists $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space equipped with a norm.

Some interesting examples of orthogonality spaces are

(a) Any real vector space X can be made into a orthogonality space with the trivial orthogonality defined on X by
  (i) for all $x \in X$, $x \perp 0$ and $0 \perp x$,
  (ii) for all $x, y \in X \setminus \{0\}$, $x \perp y$ if and only if x, y are linearly independent.
(b) Any inner product space $(X, \langle \cdot, \cdot \rangle)$ is an orthogonality space with the ordinary orthogonality given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.
(c) Any normed space $(X, \| \cdot \|)$ can be made into a orthogonality space with the Birkhoff-James orthogonality defined by $x \perp y$ if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly examples (a) and (b) are symmetric but example (c) is not. It is remarkable to note that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric.

Let X be a orthogonality vector space in the sense of Rätz and Y be an abelian group. A function $f : X \to Y$ is called *orthogonally additive*, if $f(x + y) = f(x) + f(y)$ for all $x, y \in X$ with $x \perp y$.

An orthogonally additive mapping can not be additive or linear in general. For example the orthogonally additive mapping $f : H \to \mathbb{R}$ defined on inner product space H by $f(x) = \|x\|^2$ is a quadratic function, since it satisfies the quadratic functional equation

$$q(x + y) + q(x - y) = 2q(x) + 2q(y)$$

for all $x, y \in X$.

Rätz in Corollary 7 of [1] investigated the structure of orthogonally additive mappings and showed that any orthogonally additive mapping f is of the form $a + q$, for a unique additive mapping a and a unique quadratic mapping q.

Moreover he showed that if H is a real inner product space, then any orthogonally additive mapping $f : H \to Y$ is of the form

$$f(x) = a(\|x\|^2) + b(x) \tag{1.1}$$

for all $x \in H$, where $a : \mathbb{R} \to Y$ and $b : H \to Y$ are additive mapping uniquely determined by f. In this paper, we show that any orthogonally additive isometry on an inner product space is an additive mapping.

2. The result

Theorem 2.1. *Let H be a real inner product space. If $f : H \to H$ is an orthogonally additive mapping such that*

$$\|f(x)\| = \|x\|$$

for all $x \in H$, then f is an additive mapping.

Proof. Let $\langle ., . \rangle$ denote the inner product of H. It follows from (1.1) that

$$\begin{aligned} \|x\|^2 &= \|f(x)\|^2 \\ &= \langle f(x), f(x)\rangle \\ &= \langle a(\|x\|^2) + b(x), a(\|x\|^2) + b(x)\rangle \\ &= \big\|a(\|x\|^2)\big\|^2 + 2\langle a(\|x\|^2), b(x)\rangle + \|b(x)\|^2 \end{aligned}$$

for all $x \in H$.

Let $r \in \mathbb{Q}$. Then replacing x by rx we get

$$r^2\|x\|^2 = r^4\big\|a(\|x\|^2)\big\|^2 + 2r^3\langle a(\|x\|^2), b(x)\rangle + r^2\|b(x)\|^2 \tag{2.1}$$

for all $x \in H$. Dividing the equation (2.1) by r^4 we have

$$\tfrac{1}{r^2}\|x\|^2 = \big\|a(\|x\|^2)\big\|^2 + 2\tfrac{1}{r}\langle a(\|x\|^2), b(x)\rangle + \tfrac{1}{r^2}\|b(x)\|^2$$

for all $x \in H$. Now taking limit as $r \to \infty$, we get

$$a(\|x\|^2) = 0, \quad \|b(x)\| = \|x\|$$

for all $x \in H$.

For each $t > 0$, put $x = \sqrt{t}\|y\|^{-1}y$ where $0 \neq y \in H$. Then $x \in H$ and

$$a(t) = a(t\|y\|^{-2}\|y\|^2) = a(\big\|\sqrt{t}\|y\|^{-1}y\big\|^2) = a(\|x\|^2) = 0.$$

Thus $a(t) = 0$ for all $t > 0$. Also since a is an additive mapping, so a is odd. Therefore $a(t) = -a(-t) = 0$ for all $t < 0$. This implies that $a = 0$ on $\mathbb{R}$. Thus $f(x) = b(x)$ for all $x \in H$ and f is an additive mapping. □

Proposition 2.2. *Suppose that the functions f, a and b satisfy the equation (1.1) for all $x \in H$. If $a : \mathbb{R} \to H$ and $b : H \to H$ are linear and $f : H \to H$ is bijective, then f is linear.*

Proof. Suppose that $a \neq 0$ on $\mathbb{R}$. Thus for $0 \neq a(1) \in H$, there exists a $0 \neq x_0 \in H$ such that $f(x_0) = -a(1)$. Then we have

$$-a(1) = f(x_0) = f(x) = a(\|x_0\|^2) + b(x_0) = \|x_0\|^2 a(1) + b(x_0).$$

It follows that $(1 + \|x_0\|^2)a(1) = -b(x_0)$ and Then

$$a(1) = b\Big(\frac{-x_0}{1 + \|x_0\|^2}\Big).$$

Therefore

$$f(x) = a(\|x\|^2) + b(x) = \|x\|^2 a(1) + b(x) = \|x\|^2 b\Big(\frac{-x_0}{1 + \|x_0\|^2}\Big) + b(x)$$

for all $x \in H$. So for $x = \frac{1+\|x_0\|^2}{\|x_0\|^2} x_0 \neq 0$ we have

$$\begin{aligned} f\Big(\tfrac{1+\|x_0\|^2}{\|x_0\|^2} x_0\Big) &= \Big\|\tfrac{1+\|x_0\|^2}{\|x_0\|^2} x_0\Big\|^2 b\Big(\frac{-x_0}{1+\|x_0\|^2}\Big) + b\Big(\tfrac{1+\|x_0\|^2}{\|x_0\|^2} x_0\Big) \\ &= b\Big(\Big\|\tfrac{1+\|x_0\|^2}{\|x_0\|^2} x_0\Big\|^2 \frac{-x_0}{1+\|x_0\|^2} + \tfrac{1+\|x_0\|^2}{\|x_0\|^2} x_0\Big) \\ &= b\Big(-\tfrac{1+\|x_0\|^2}{\|x_0\|^2} x_0 + \tfrac{1+\|x_0\|^2}{\|x_0\|^2} x_0\Big) = b(0) = 0. \end{aligned}$$

This contradicts the injectivity of f. Thus $a = 0$ on $\mathbb{R}$ and then $f = b$ is linear. $\square$

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FIXED POINT THEOREMS FOR CYCLIC WEAK CONTRACTIONS IN MODULAR METRIC SPACES

HOSSEIN RAHIMPOOR

Department of Mathematics, Payame Noor University P. O. Box 19395-3697, Tehran, Iran
rahimpoor2000@pnu.ac.ir

Abstract. The purpose of this paper is to present some fixed point results for ϕ-contractions in modular metric spaces.

1. Introduction

The concept of modular spaces was introduced by Nakano [10] and was later reconsidered in detail by Musielak and Orlicz [8, 9]. In 2010, Chistyakov [2] introduced a new metric structure, which has a physical interpretation and generalized modular spaces and complete metric spaces by introducing modular metric spaces. For more features of concepts of modular metric spaces, see e. g., [1, 3, 4]. Fixed point theory involves many fields of mathematics and branches of applied science such as functional analysis, mathematical analysis, general topology and operator theory. In 2003, Kirk et al. [7] introduced cyclic contraction in metric spaces and investigated the existence of proximity points and fixed points for cyclic contraction mapping. Later, Karapinar and Erhan[6] proved the existence of fixed points for various types of cyclic contractions in a metric space. Recently, E. Karapinar in [5] proves a fixed point theorem for an operator T on a complete metric space X when X has a cyclic representation with respect to T. In this paper, we improve and generalized the fixed point results for mappings satisfying cyclical contractive conditions established by E. Karapinar [5], in modular metric spaces.

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Definition 1.1. Let X be an arbitrary set, the function $\omega : (0,\infty) \times X \times X \longrightarrow [0,\infty]$ that will be written as $\omega_\lambda(x,y) = \omega(\lambda,x,y)$ for all $x,y \in X$ and for all $\lambda > 0$, is said to be a modular metric on X (or simply a modular if no ambiguity arises) if it satisfies the following three conditions:
(i) given $x,y \in X$, $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$ if and nonly if $x = y$;
(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$, for all $\lambda > 0$ and $x,y \in X$;
(iii) $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda,\mu > 0$ and $x,y,z \in X$.
If instead of (i), we have only the condition:
(i_1) $\omega_\lambda(x,x) = 0$ for all $\lambda > 0$ and $x \in X$, then ω is said to be a (metric) pseudomodular on X and if ω satisfies (i_1) and
(i_2) given $x,y \in X$, if there exists $\lambda > 0$, possibly depending on x and y, such that $\omega_\lambda(x,y) = 0$ implies that $x = y$, then ω is called a *strict modular* on X.

If instead of (iii) we replace the following condition for all $\lambda,\mu > 0$ and $x,y,z \in X$;

$$\omega_{\lambda+\mu}(x,y) \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\lambda(z,y), \tag{1.1}$$

then ω is called a *convex modular* on X.

Definition 1.2. [2] Given a modular ω on X, the sets

$$X_\omega \equiv X_\omega(x_\circ) = \{x \in X : \omega_\lambda(x,x_\circ) \to 0 \ as\ \lambda \to \infty\}$$

and

$$X^*_\omega \equiv X^*_\omega(x_\circ) = \{x \in X : \omega_\lambda(x,x_\circ) < \infty \ \ for\ some\ \ \lambda > 0\}$$

are said to be modular spaces (around $x_\circ$). Also the modular spaces X_ω and X^*_ω can be equipped with metrics d_ω and d^*_ω, generated by ω and given by

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leq \lambda\}, \quad x,y \in X_\omega$$

and

$$d^*_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leq 1\}, \quad x,y \in X^*_\omega$$

If ω is a convex modular on X, then according to [2, Theorem 3.6] the two modular spaces coincide, $X_\omega = X^*_\omega$.

Definition 1.3. Given a modular metric space X_ω, a sequence of elements $\{x_n\}_{n=1}^\infty$ from X_ω is said to be modular convergent ($\omega-$convergent) to an element $x \in X$ if there exists a number $\lambda > 0$, possibly depending on $\{x_n\}$ and x , such that $\lim_{n\to\infty}\omega_\lambda(x_n,x) = 0$. This will be written briefly as $x_n \overset{\omega}{\to} x$, as $n \to \infty$.

Definition 1.4. [4] A sequence $\{x_n\} \subset X_\omega$ is said to be ω-*Cauchy* if there exists a number $\lambda = \lambda(\{x_n\}) > 0$ such that $\lim_{m,n\to\infty}\omega_\lambda(x_n,x_m) = 0$, i.e.,

$$\forall\varepsilon > 0 \ \ \exists \ \ n_\circ(\varepsilon) \in \mathbb{N} \ \ such \ \ that \ \ \forall n,m \geq n_\circ(\varepsilon) \ \ : \ \ \omega_\lambda(x_n,x_m) \leq \varepsilon.$$

Modular metric space X_ω is said to be ω-*complete* if each ω-Cauchy sequence from X_ω be modular convergent to an $x \in X_\omega$.

Remark 1.5. A modular $\omega = \omega_\lambda$ on a set X, for given $x, y \in X$, is non-increasing on λ. Indeed if $0 < \lambda < \mu$, then we have

$$\omega_\mu(x,y) \le \omega_{\mu-\lambda}(x,x) + \omega_\lambda(x,y) = \omega_\lambda(x,y)$$

for all $x, y \in X$.

2. Main result

Definition 2.1. Let X_ω be a modular metric space, $p \in \mathbb{N}$, and $T : X_\omega \to X_\omega$ a map. Then we say that $\cup_{i=1}^{p} A_i$ (where $A_i \subseteq X_\omega$ for all $i \in \{1, 2, ..., p\}$) is a cyclic representation of X with respect to T if and only if the following two conditions hold:
(I) $X_\omega = \cup_{i=1}^{p} A_i$;
(II) $T(A_i) \subseteq A_{i+1}$ for $1 \le i \le p-1$, and $T(A_p) \subseteq A_1$.

Definition 2.2. Let X_ω be a modular metric space, m a positive integer, $A_1, A_2, ..., A_m$ ω-closed nonempty subset of $X_\omega$and $Y = \cup_{i=1}^{m} A_i$ and $T : Y \to Y$ an operator. T is called a cyclic weak ϕ-contraction if
(I) $\cup_{i=1}^{m} A_i$is a cyclic representation of Y with respect to T;
(II) there exists a non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) > 0$ for $t \in (0,\infty)$ and $\phi(0) = 0$, such that

$$\omega_\lambda(Tx, Ty) \le \omega_\lambda(x,y) - \phi(\omega_\lambda(x,y)) \tag{2.1}$$

for all $\lambda > 0$ and for any $x \in A_i, y \in A_{i+1}$, $i = 1, 2, ..., m$ where $A_{m+1} = A_1$.

Example 2.3. Let $X_\omega = [0,1]$, we take a mapping $\omega : (0,\infty) \times [0,1] \times [0,1] \to [0,\infty]$ which is defined by $\omega_\lambda(x,y) = \dfrac{|x-y|}{\lambda}$ for all $x, y \in X = X_\omega$ and $\lambda > 0$. Consider the ω-closed nonempty subsets of X_ω as follow : $A_1 = [0,1], A_2 = [0, \frac{2}{3}], A_3 = [0, \frac{1}{2}], A_4 = [0, \frac{5}{12}], A_5 = [0, \frac{3}{8}]$ with $X_\omega = Y = \cup_{i=1}^{5} A_i$. Let $T : X_\omega \to X_\omega$ be the mapping defined by $Tx = \dfrac{3x+1}{6}$. Then, $T(A_1) \subseteq A_2, T(A_2) \subseteq A_3, T(A_3) \subseteq A_4, T(A_4) \subseteq A_5, T(A_5) \subseteq A_1$. And

$$\omega_\lambda(Tx,Ty) = \frac{|\frac{3x+1}{6} - \frac{3y+1}{6}|}{\lambda} = \frac{1}{\lambda}(\frac{|x-y|}{2}) \le \omega_\lambda(x,y) - \frac{1}{2}\omega_\lambda(x,y).$$

Furthermore, if $\phi : [0,\infty) \to [0,\infty)$ is defined by $\varphi(t) = \frac{t}{2}$, then ϕ is strictly increasing and T is a cyclic weak ϕ-contraction.

Remark 2.4. Rewriting the inequality 1.1 in the form

$$(\lambda + \mu)\omega_{\lambda+\mu}(x,y) \le \lambda\omega_\lambda(x,z) + \mu\omega_\mu(y,z)$$

we find that the function ω is a convex modular on X if and only if the function $\widehat{\omega}(x,y) = \lambda\omega_\lambda(x,y)$ for all $\lambda > 0$ and $x, y \in X$, is simply a modular on X, and the function $\lambda \mapsto \widehat{\omega}(x,y) = \lambda\omega_\lambda(x,y)$ are non-increasing on $(0,\infty)$:

$$\text{if } 0 < \lambda \le \mu, \text{ then } \omega_\mu(x,y) \le \frac{\lambda}{\mu}\omega_\lambda(x,y) \le \omega_\mu(x,y).$$

Now, for any $\mu \geq \lambda$ we find $k \in \mathbb{R}_+$ such that $\mu = k\lambda$ and so

$$\omega_{k\lambda}(x, y) \leq \frac{1}{k}\omega_\lambda(x, y). \tag{2.2}$$

Theorem 2.5. *Let ω be a convex modular on X such that X_ω is a $\omega-$complete modular metric space, m is a positive integer, $A_1, A_2, ..., A_m$ ω-closed subsets of X_ω and $Y = \cup_{i=1}^m A_i$. Suppose that $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t)$ is a non-decreasing function and $\varphi(t) = 0$ only for $t = 0$ and $T : X_\omega \times X_\omega \to X_\omega$ is a cyclic weak φ-contraction where $Y = \cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T. Then, T has a unique fixed point $z \in \cap_{i=1}^m A_i$.*

Theorem 2.6. *Let $T : Y \to Y$ be a self mapping as in Theorem 2.5 .*
(i) If there exists a sequence $\{y_n\}$ in Y with $\lim_{n\to\infty} \omega_\lambda(y_n, Ty_n) = 0$ then $\lim_{n\to\infty} \omega_\lambda(y_n, z) = 0$.
(ii) If there exists a ω-convergent sequence $\{y_n\}$ in Y with $\lim_{n\to\infty} \omega_\lambda(y_{n+1}, Ty_n) = 0$ then there exists $x \in Y$ such that $\lim_{n\to\infty} \omega_\lambda(y_n, T^n x) = 0$.

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مجموعه مقاله های فارسی

۷ امین سمینار آنالیز تابعی و کاربردهای آن
۱۰ و ۱۱ اسفند ۱۴۰۱
دانشگاه بین المللی امام خمینی (ره)
قزوین، ایران
IMAM KHOMEINI INTERNATIONAL UNIVERSITY
ISC

سخنرانی

یک کران بالا برای مقدار ویژه اصلی P-Q -لاپلاسین

مهدی لطیفی [1]

[1] گروه ریاضی، دانشکده علوم پایه، دانشگاه پدافند هوایی خاتم الانبیاء(ص)
m_ latifi@khadu.ac.ir

چکیده. در این پژوهش برای مقدار ویژه اصلی مسئله مقدار ویژه غیر خطی $p-q$-لاپلاسین با شرایط مرزی دیریکله بر روی یک ناحیه کراندار در $\mathbb{R}^N$، یک کران بالا ارائه می دهیم. هرچند این کران بالا بهترین نمی باشد اما روش بکار گرفته شده جدید می باشد.

۱. پیش‌گفتار

خواص مقادیر ویژه عملگرها و بخصوص عملگرهای خانواده لاپلاسین، یکی از موضوعات مورد توجه محققین در دهه های اخیر می باشد[۲]. روش های تغییراتی ابزاری مناسب برای حصول نتایجی در زمینه وجود و رفتار مقادیر ویژه می باشد[۳، ۱]. در این پژوهش مسئله مقدار ویژه زیر را مد نظر قرار می دهیم:

$$D(\Omega):\begin{cases}-\Delta_p u-\Delta_q u=\lambda|u|^{p-2}u, & \text{در } \Omega\text{؛}\\ u=0, & \text{روی } \partial\Omega\text{؛}\end{cases} \qquad (۱.۱)$$

که Ω یک ناحیه کراندار در $\mathbb{R}^N$، $\Delta_p u:=div(|\nabla u|^{p-2}\nabla u)$ عملگر p-لاپلاسین با $p>1$ و $1=\frac{1}{p}+\frac{1}{q}$ است. وجود مقدار ویژه برای این مسئله با استفاده از اصل Ljusternik–Schnirelman و بررسی شرایط آن نتیجه می شود.

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واژگان کلیدی. مقدار ویژه اصلی، نامساوی پوانکاره ، اصل Ljusternik–Schnirelman .

اگر X یک فضای باناخ حقیقی و انعکاسی با بعد نامتناهی، و F, G دو تابع روی X باشند. برای ثابت $\alpha > 0$، مسئله مقدار ویژه

$$F'(u) = \lambda G'(u) \qquad u \in N_\alpha, \lambda \in \mathbb{R} \qquad (۲.۱)$$

که $N_\alpha := \{u \in X; G(u) = \alpha\}$ مجموعه تراز G است و

الف) تابعک های $F, G : X \to \mathbb{R}$، تابعک هایی زوج در $C^1(X, \mathbb{R})$ باشند.

ب) F' پیوسته قوی باشد و از $F(u) \neq 0$ برای $u \in \overline{co}N_\alpha$، نتیجه شود $F'(u) \neq 0$.

ج) عملگر G' روی مجموعه های کراندار، پیوسته یکنواخت باشد و در شرط $(S)_1$ صدق کند.

د) مجموعه تراز N_α کراندار باشد و برای $u \neq 0$ داشته باشیم:

$$\langle G'(u), u\rangle > 0, \lim_{t\to+\infty} G(tu) = +\infty, \quad \inf_{u\in N_\alpha} \langle G'(u), u\rangle \ > 0$$

برای هر $n \in \mathbb{N}$، $\mathcal{A}_n$ را خانواده همه زیرمجموعه های متقارن و فشرده K از N_α که $gen(K) \geq n$ باشد و $F(u) > 0$ می گیریم، و تعریف می کنیم:

$$\pm c_n = \begin{cases} \sup_{K\in\mathcal{A}_n} \inf_{u\in K} \pm F(u) & \text{اگر } \mathcal{A}_n \neq \emptyset \\ 0 & \text{اگر } \mathcal{A}_n = \emptyset \end{cases}$$

برای $n = 1, 2, ...$ و

$$\chi_\pm := \begin{cases} \sup\{n \in \mathbb{N} : \pm c_n > 0\} & \text{اگر } c_1 > 0 \\ 0 & \text{اگر } c_1 = 0 \end{cases}$$

قضیه ۱.۱. (Ljusternik–Schnirelman)[۴] با فرض برقراری شرایط (الف–د) احکام زیر برقرار است:

۱) وجود مقادیر ویژه: اگر $\pm c_n > 0$ ، (+ یا −) در آن صورت (۲.۱) دارای زوج $(u_n, -u_n)$ با مقدار ویژه $\lambda_n \neq 0$ و $F(u_n) = c_n$ است.
۲) چندگانگی: (۲.۱) ، حداقل دارای $\chi_+ + \chi_-$ زوج $(u, -u)$ از توابع ویژه با مقادیر ویژه مخالف صفر است.

۳) سطوح بحرانی: $\pm\infty > \pm c_1 \geq \pm c_2 \geq ... \geq 0$ و $c_n \to 0$ بگونه ای که $n \to \infty$.

۴) نامتناهی مقدار ویژه: اگر $\chi_+ = \infty$ یا $\chi_- = \infty$ و $F(u) = 0$ ، برای $u \in \overline{co}N_\alpha$ نتیجه دهد که $\langle F'(u), u\rangle = 0$، در آن صورت دنباله نامتناهی (λ_n) از مقادیر ویژه متمایز برای (۲.۱) وجود دارد که $\lambda_n \to 0$.
۵) همگرایی ضعیف توابع ویژه: اگر از $F(u) = 0, u \in \overline{co}N_\alpha$ نتیجه شود، $u = 0$ در آن صورت $\max(\chi_+, \chi_-) = +\infty$ و یک دنباله (u_n, λ_n) از جواب های ویژه (۲.۱) وجود دارد که $u_n \rightharpoonup 0, \lambda_n \to 0$ و $\lambda_n \neq 0$ برای هر n .

۲. کران بالا برای λ_1

گزاره ۱.۲. (نامساوی پوانکاره در $(\dot{W}_{1,p}(\Omega)$) فرض کنید مجموعه باز $\Omega \subset \mathbb{R}^N$ دارای عرض متناهی d باشد، یعنی بین دو ابر صفحه موازی با فاصله d قرار گیرد، و $1 \leq p < \infty$. در آن صورت برای هر داریم:

$$\int_\Omega |u(x)|^p dx \leq \frac{d^p}{p} \int_\Omega |\nabla u(x)|^p dx.$$

برهان. بدون کاسته شدن از کلیت فرض می کنیم Ω بین ابر صفحه های $x_N = 0$ و $x_N = d$ قرار دارد. لذا برای $u \in C_c^\infty(\Omega)$ داریم:

$$\begin{aligned} |u(x', x_N)| = |u(x', x_N) - u(x', 0)| &= \left| \int_0^{x_N} \frac{\partial u}{\partial x_N}(x', t) dt \right| \\ &\leq x_N^{\frac{1}{q}} \left(\int_0^d \left| \frac{\partial u}{\partial x_N}(x', t) \right|^p dt \right)^{\frac{1}{p}}. \end{aligned}$$

با استفاده از قضیه تونیلی بدست می آوریم:

$$\begin{aligned} \int_{\mathbb{R}^{N-1}\times[0,d]} |u(x', x_N)|^p \, dx &\leq \int_{\mathbb{R}^{N-1}} \int_0^d x_N^{\frac{p}{q}} \int_0^d \left| \frac{\partial u}{\partial x_N}(x', t) \right|^p dt dx_N dx' \\ &= \int_\Omega \left| \frac{\partial u}{\partial x_N}(y) \right|^p dy \int_0^d x_N^{p-1} dx_N = \frac{|d|^p}{p} \int_\Omega \left| \frac{\partial u}{\partial x_N}(y) \right|^p dy \\ &\leq \frac{|d|^p}{p} \int_\Omega |\nabla u(x)|^p \, dx. \end{aligned}$$

□

اکنون با توجه به تعریف c_n و بکار گیری نامساوی فوق می توانیم کران بالایی برای اولین مقدار ویژه مسئله بدست آوریم.

گزاره ۲.۲. با فرض $q \geq p$ برای مقدار ویژه اصلی مسئله (۱.۱) داریم:

$$\lambda_1 \leq \frac{d^p}{q}.$$

برهان. اگر K زیرمجموعه ای متقارن و فشرده از N_α باشد، $c_1 = \sup_{K\in\mathcal{A}_1}\inf_{u\in K} F(u)$ ، اما با نامساوی پوانکاره برای هر $u \in \dot{W}_{1,p}(\Omega)$:

$$\begin{aligned} F(u) = \int_\Omega \frac{|u(x)|^p}{p}dx \leq \frac{1}{p}\left(\frac{d^p}{p}\int_\Omega |\nabla u|^p\right) \\ \leq \frac{1}{p}\left(\frac{d^p}{p}\int_\Omega |\nabla u|^p + \frac{d^p}{q}\int_\Omega |\nabla u|^q\right) \\ = \frac{d^p\alpha}{p}, \end{aligned}$$

بنابراین $c_1 \leq \frac{d^p\alpha}{p}$. همچنین اگر

$$F(u) = \int_\Omega \frac{|u|^p}{p},\ G_1(u) = \int_\Omega \frac{|\nabla u|^p}{p},\ G_2(u) = \int_\Omega \frac{|\nabla u|^q}{q},$$

دیده می شود که F', G_1', G_2' توابعی مثبت و همگن بترتیب از درجه $p-1, p-1, q-1$ هستند. بنابراین

$$F(u) = \int_0^1 \langle F'(tu), u\rangle\, dt = \frac{\langle F'(u), u\rangle}{p}$$

و به طور مشابه برای توابع $G_1(u), G_2(u)$. اکنون اگر (u_1, λ_1) ، یک مقدار ویژه از (۱.۱) با $G(u_1) = \alpha$ باشد ، داریم:

$$\begin{aligned} F'(u_1) &= \lambda_1 \left[G_1'(u_1) + G_2'(u_1)\right] \ \Rightarrow \\ \langle F'(u_1), u_1\rangle &= \lambda_1 \left[\langle G_1'(u_1), u_1\rangle + \langle G_2'(u_1), u_1\rangle\right] \ \Rightarrow \\ \lambda_1 &= \frac{\langle F'(u_1), u_1\rangle}{\langle G_1'(u_1), u_1\rangle + \langle G_2'(u_1), u_1\rangle} = \frac{pF(u_1)}{pG_1(u_1) + qG_2(u_1)} \\ &\leq \frac{p}{q}\frac{c_1}{\alpha}, \end{aligned}$$

با توجه به اینکه $c_1 \leq \frac{d^p}{p}$ بدست می آوریم $\lambda_1 \leq \frac{d^p}{q}$. □

برای حالت خاص $q = p = 2$ بدست می آوریم $\lambda_1 \leq \frac{d^2}{2}$ ، که یک نتیجه مناسب است.

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قزوین، ایران

سخنرانی *:سخنران

حل عددی مسائل الاستیسیته غیرخطی در حساب تغییرات

مژگان تقوی [1] * و محمدصادق شاهرخی دهکردی [2]

[1] گروه ریاضی کاربردی و صنعتی، دانشکده علوم ریاضی، دانشگاه شهید بهشتی
mo_taghavi@sbu.ac.ir
[2] گروه ریاضی کاربردی و صنعتی، دانشکده علوم ریاضی، دانشگاه شهید بهشتی
M_Shahrokhi@sbu.ac.ir

چکیده. در این مقاله با استفاده از روش عددی، تقریبی از مینیمم‌کننده‌های مسائل تغییراتی در الاستسیته غیرخطی بدست می‌آوریم. برای این منظور معادلات اویلر-لاگرانژ مربوط به تابعک انرژی را به وسیله روش طیفی گسسته سازی کرده و دستگاه جبری بدست آمده را حل می کنیم. در آخر با استفاده از نتایج عددی کارایی روش موردنظر را نشان می دهیم.

۱. پیش‌گفتار

فرض کنید Ω یک دامنه کراندار باشد. مساله تغییراتی زیر را در نظر بگیرید

$$\mathbb{F}[u,\Omega] := \int_{\Omega} W(\nabla u(x))dx. \tag{۱.۱}$$

تابعک انرژی بالا مربوط به مسائل الاستیسیته غیرخطی می‌باشند. ما به دنبال یافتن مینیمم کننده این تابعک روی مجموعه ای از توابع که به فرم

$$\mathcal{A}_p(\Omega) := \Big\{ u \in W^{1,p}(\Omega,\mathbb{R}^n) : u|_{\partial\Omega} = \varphi(x), \det\nabla u > 0 \ \text{-a.e} \Big\},$$

تعریف می شوند، هستیم. در تابعک انرژی (۱.۱) ، $W(\nabla u(x)) : M_+^{n\times n} \to \mathbb{R}^+$ تابع چگالی انرژی ذخیره شده است، که در آن $M^{n\times n}$ مجموعه ای از ماتریس های $n\times n$ با دترمینان

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* سخنران

مثبت هستند و در شرط مرزی $u = \varphi(x)$ صدق می کنند. هم چنین $h \in C^2(0, \infty)$ یک تابع محدب است که در شرایط زیر برای آن برقرار است:

[**۱H**] $h : (0, \infty) \to (0, \infty)$

[**۲H**] $\lim_{t \downarrow 0} h(t) = \lim_{t \uparrow \infty} \frac{h(t)}{t} = \infty$

در اینجا $\Omega \subset \mathbb{R}^2$ را یک مجموعه‌ی باز و کراندار به صورت $\{x \in \mathbb{R}^2 | a < |x| < b\}$ درنظر می گیریم که $0 < a < b < \infty$ است. در مقاله‌های [۱، ۵، ۲، ۳] از روش عناصرمتناهی با شبکه بندی‌های متفاوت برای حل عددی مسائل تغییراتی (۱.۱) که دارای یک یا چند حفره هستند استفاده شده‌است. هم‌چنین همگرایی، کارآمدی و موثر بودن این روش‌ها بررسی شده است. تحلیل و بررسی نتایج عددی در این مقالات ما را به استفاده از روش های عددی برای بدست آوردن تقریبی از مینیمم کننده های مسائل تغییراتی دیگر ترغیب کرد. در این مقاله تابع چگالی انرژی موردنظر از مدل اسکرم [۴] بدست می آید که تاکنون روش عددی برای آن مورد استفاده قرار نگرفته است و به صورت زیر بیان می شود.

$$W(\nabla u(x)) := | \wedge^2 \nabla u|^p + h(\det \nabla u). \tag{۲.۱}$$

در ادامه مینیمم‌کننده‌های این مساله را با استفاده از روش طیفی تقریب می زنیم که برای این‌کار ابتدا با استفاده از مقادیر منفرد ماتریس ∇u برای تابع چگالی (۲.۱) ، مساله تغییراتی (۱.۱) را به فرم دیگری بازنویسی می‌کنیم.

۲. دست‌آوردهای پژوهش

در این بخش روش عددی موردنظر را بر روی دستگاه معادلات اویلر-لاگرانژ که به فرم قطبی بیان شده‌اند، پیاده‌سازی می‌کنیم. حال فرم قطبی مساله تغییراتی که به صورت

$$\begin{aligned} \mathbb{E}[u, \Omega] &= \int_{\tilde{\Omega}} \Big(F(g(P, Q)) + h(d((P, Q))\Big) \mu(\rho) d\rho d\varphi \\ &=: \mathbb{F}[P, Q] \end{aligned} \tag{۱.۲}$$

می‌باشد را درنظر بگیرید که در آن $\mu(\rho) := \dfrac{(b^2 - a^2) + (b - a)^2 \rho}{4}$ و $\tilde{\Omega}$ فرم استاندارد Ω می‌باشد. معادلات اویلر-لاگرانژ را نسبت به متغیرهای (P, Q) بدست می‌آوریم. توابع هموار $(\bar{P}, \bar{Q})$ که روی مرز صفر هستند را در نظر بگیرید. حال معادلات اویلر-لاگرانژ تابعک انرژی $\mathbb{F}$ به فرم زیر محاسبه می‌شوند:

$$\int_{\tilde{\Omega}} f_1(P, Q, \bar{P}) d\rho d\varphi = 0, \qquad \int_{\tilde{\Omega}} f_2(P, Q, \bar{Q}) d\rho d\varphi = 0. \tag{۲.۲}$$

توجه کنید که در آن f_1 و f_2 به صورت

$$f_1 = \bar{P}\Big\{\frac{4}{\mu(\rho)}F'(g)P\Big[Q_\rho^2\big(P_\varphi^2 + P^2(Q_\varphi+1)^2\big)+ (Q_\varphi+1)^2(P_\rho^2 + P^2\psi_\rho^2) - Q_\rho(Q_\varphi+1)\Big] + h'(d)P_\rho\times (Q_\varphi+1)\Big\} + (\bar{P})_\rho\Big\{\frac{4}{\mu(\rho)}F'(g)P^2\Big[P_\rho(Q_\varphi+1)- P_\varphi Q_\varphi)(Q_\varphi+1)\Big] + h'(d)P(Q_\varphi+1)\Big\} - (\bar{P})_\varphi\Big\{\frac{4}{\mu(\rho)}\times F'(g)\Big[P_\varphi(P_\rho^2 + P^2\psi_\rho^2) - P_\rho\Big] + h'(d)Q_\rho\Big\},$$

و

$$f_2 = (\bar{Q})_\rho\Big\{\frac{2}{\mu(\rho)}F'(g)\Big[2P^2\psi_\rho\big(P_\varphi^2 + P^2(Q_\varphi+1)^2\big)- P^2(Q_\varphi+1)\Big] - h'(d)P_\varphi P\Big\} + (\bar{Q})_a\Big\{\frac{2}{\mu(\rho)}F'(g)\times \Big[2\Gamma^2(Q_\varphi+1)(P_\rho^2 + P^2\psi_\rho^2) - P^2\psi_\rho\Big] + h'(d)PP_\rho\Big\},$$

بدست می آیند. در این قسمت روش طیفی را بر روی معادلات اویلر-لاگرانژ (۲.۲) پیاده سازی می کنیم. برای این منظور تابع $P(\rho,\varphi)$ به صورت

$$P^{N,M} := P^{N,M}(\rho,\varphi) = \sum_{j=0}^{M}\Bigg[\sum_{i=0}^{\frac{N}{2}}\alpha_{i,j}\cos(i\varphi) + \sum_{i=1}^{\frac{N}{2}-1}\beta_{i,j}\sin(i\varphi)\Bigg]l_j(\rho), \qquad (۳.۲)$$

و به طور مشابه تابع $Q(\rho,\varphi)$ را به فرم

$$Q^{N,M} := Q^{N,M}(\rho,\varphi) = \sum_{j=0}^{M}\Bigg[\sum_{i=0}^{\frac{N}{2}}\xi_{i,j}\cos(i\varphi) + \sum_{i=1}^{\frac{N}{2}-1}\eta_{i,j}\sin(i\varphi)\Bigg]l_j(\rho), \qquad (۴.۲)$$

بیان می کنیم، که در آن $l_j(\rho)$ چندجمله‌ای لاگرانژ می باشد. در گام بعدی با استفاده از روش های انتگرال‌گیری عددی مناسب و اعمال شرایط مرزی بر آن یک دستگاه معادلات جبری بدست می‌آوریم که با حل آن تقریبی از مجهولات $(P^{N,M}, Q^{N,M})$ بدست می آید. در ادامه با بیان یک مثال کارایی روش ارائه شده را بررسی می کنیم.

قضیه ۱.۲. مساله تغییراتی $\mathbb{F}[P^{N,M}, Q^{N,M}]$ را درنظر بگیرید. اگر $(M,N) \to +\infty$ آن‌گاه تابعک انرژی $\mathbb{F}[P^{N,M}, Q^{N,M}]$ به $\mathbb{F}[P,Q]$ همگرا است.

مثال ۲.۲. در تابع چگالی (۲.۱) فرض کنید $p = \frac{3}{2}$ و در دامنه Ω، $b = 1$ باشد. همچنین تابع محدب $h(t) = 2^{-\frac{1}{4}}\left(\frac{(t-1)^2}{2} + \frac{1}{t}\right)$ را درنظر می‌گیریم. جدول زیر خطای تابعک انرژی (۱.۲) را با استفاده از روش ارائه شده برای $a = 10^3$ نشان می دهد.

جدول ۱: خطای تابعک انرژی $\mathbb{F}[\cdot;\tilde{\Omega}]$ با $a = 10^{-3}$

M	24	32	40	48
$a = 10^{-3}$	$1.7862e-04$	$3.965e-04$	$8.8149e-04$	$9.4425e-05$

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قزوین، ایران

مینیمم‌کننده‌های موضعی در حساب تغییرات کسری

فاطمه طوس‌نژاد [1] * و محمد صادق شاهرخی دهکردی [2]

[1] گروه ریاضی کاربردی و صنعتی، دانشکده علوم ریاضی، دانشگاه شهید بهشتی
f_toosnezhad@sbu.ac.ir
[2] گروه ریاضی کاربردی و صنعتی ، دانشکده علوم ریاضی، دانشگاه شهید بهشتی
m_ shahrokhi@sbu.ac.ir

چکیده. حساب تغییرات کسری یکی از شاخه‌های مهم ریاضیات می‌باشد، که در سال های اخیر بسیار مورد توجه ریاضیدانان قرارگرفته است. ما در این مقاله قصد داریم، اکسترمم‌های موضعی تابعی انرژی را برای حالتی که تابعی شامل مشتقات کسری از نوع کاپوتو باشد، در بعد دو به کمک روش مستقیم و معادله اولر–لاگرانژ مورد بررسی قرار دهیم.

۱. پیشگفتار

فرض کنید $\Omega = \{x \in \mathbb{R}^2 : a < |x| < b\}$ دامنه‌ای با مرز لیپ شیتز باشد، تابعی انرژی

$$\mathbb{E}[u, \Omega)] = \int_{\Omega} \frac{1}{2}|\nabla u|^2 + \phi(\det \nabla u) dx,$$

را روی فضای توابع قابل قبول

$$\mathcal{A}(\Omega) = \left\{ u \in W^{1,2}(\Omega, \mathbb{R}^2) : u|_{\partial\Omega} = x, \det \nabla u > 0 \text{ a.e. in } \Omega \right\},$$

در نظر بگیرید. همچنین $\phi : (0, \infty) \to (0, \infty)$ که در ضابطه‌ی تابعی حضور دارد، یک تابع محدب است و در فضای $C^2(0, \infty)$ قرار دارد.

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واژگان کلیدی. حساب تغییرات کسری، روش مستقیم، مشتق کسری کاپوتو، ،معادله اولر– لاگرانژ .

* سخنران

تابع u را به فرم کلی مختصات قطبی

$$(r,\theta) \longmapsto (\rho(r,\theta), \varphi(r,\theta))$$

انتقال می‌دهیم، سپس مشتقات جزیی مرتبه‌ی صحیح را به مشتقات کسری جزیی کاپوتو از مرتبه‌ی $0 < \alpha < 1$ تبدیل می‌کنیم(به منظور مطالعه در زمینه حساب کسری مرجع [۲] پیشنهاد می‌شود). اگر تابع u در شرایط زیر صدق کند، آن‌گاه در فضای توابع $\mathcal{A}(\Omega)$ قرار می‌گیرد

$$\begin{aligned} &\rho(r,\theta) = \rho(r), \;\; \rho(a) = a, \;\; \rho(b) = b, \;\; {}_a^C\mathcal{D}_r^\alpha \rho > 0 \\ &\varphi(r,\theta) = \theta + g(r) \end{aligned}$$

به طوری که $g \in W^{1,2}(a,b)$. با اعمال این شرایط، تابعی انرژی به فرم زیر تبدیل می‌شود
(۱.۱)

$$\mathbb{E}(\rho,g) = \int_a^b \left(\pi\big(({}_a^C\mathcal{D}_r^\alpha \rho)^2 + \rho^2(({}_a^C\mathcal{D}_r^\alpha g)^2 + \frac{c}{r^2})\big) + \Phi(\frac{\rho}{r}{}_a^C\mathcal{D}_r^\alpha \rho)\right) r dr,$$

که در آن تابع Φ ویژگیهای تابع ϕ را به ارث می‌برد و ثابت c وابسته به α همواره مثبت است. هدف ما در این مقاله تعیین مینیمم‌های تابعی انرژی $\mathbb{E}$ روی فضای توابع قابل قبول زیر می‌باشد

$$\mathfrak{B} = \left\{ \begin{array}{l} (\rho,g,\alpha): \\ 0 < \alpha < 1, \\ \rho \in W^{1,2}[a,b], \\ \rho(a) = a, \; \rho(b) = b, \; {}_a^C\mathcal{D}_r^\alpha \rho > 0 \; a.e, \\ {}_a^C\mathcal{D}_r^\alpha \rho, \; {}_r^C\mathcal{D}_b^\alpha g \in L^2[a,b], \\ g \in W^{1,2}[a,b], \; {}_a^C\mathcal{D}_r^\alpha g, \; {}_r^C\mathcal{D}_b^\alpha g \in L^2[a,b]. \end{array} \right\}$$

اگر $\alpha = 1$ آن‌گاه این مسئله، حالت خاصی از مسئله مطرح شده‌ی مقاله [۳] می‌باشد، به طوری که مینیمم‌کننده‌هایی برای تابعی از کلاس خاصی از توابع با عنوان تاب‌های تعمیم یافته موجود است.

۲. دست‌آوردهای پژوهش

یکی از راه های پیدا کردن اکسترمم‌های تابعی بررسی جواب‌های معادلات اولر – لاگرانژ مربوط به تابعی می‌باشد، لذا در این بخش ابتدا معادلات اولر–لاگرانژ تابعی $\mathbb{E}$ را روی فضای توابع قابل قبول $\mathfrak{B}$ بدست می‌آوریم. سپس به کمک روش مستقیم نشان می‌دهیم مینیمم‌کننده‌های تابعی روی هر کدام از کلاس‌های خاص هموتوپی وجود دارد. این کلاس‌ها فضای توابع مفروض را افراز می‌کنند.

قضیه ۱.۲. فرض کنید $(\rho,g) \in \mathfrak{B}$ به طوری که ${}_a^C\mathcal{D}_r^\alpha g \in C^1(a,b)$ ، $\rho \in C^1(a,b)$ و ${}_a^C\mathcal{D}_r^\alpha \rho \in C^1(a,b)$. همچنین با فرض $\mathbb{E}(\rho,g) < \infty$، معادلات اولر–لاگرانژ محدود به $\mathbb{E}$

روی فضای توابع مفروض $\mathfrak{B}$ در(ρ, g) به شکل زیر می‌شود

$$\begin{aligned} &{}_r\mathcal{D}_b^\alpha(\rho^2 r\, {}_a^C\mathcal{D}_r^\alpha g) = 0 \\ -&{}_r\mathcal{D}_b^\alpha(2\pi r\, {}_a^C\mathcal{D}_r^\alpha \rho + \rho\Phi'(\frac{\rho}{r}\, {}_a^C\mathcal{D}_r^\alpha \rho)) = 2\pi\rho(r\, ({}_a^C\mathcal{D}_r^\alpha g)^2 + \frac{c}{r}) \\ &\qquad + {}_a^C\mathcal{D}_r^\alpha \rho\, \Phi'(\frac{\rho}{r}\, {}_a^C\mathcal{D}_r^\alpha \rho) \end{aligned} \tag{۱.۲}$$

برهان. از آن جایی که (ρ, g) تابعی ۱.۱ را مینیمم می‌کنند، لذا برای $\varepsilon > 0$ دلخواه و تابع $h \in C^1[(a,b), \mathbb{R}^2]$ که مقدار آن روی مرز (a,b) صفر می‌شود، داریم

$$\frac{d}{d\varepsilon}\mathbb{E}(\rho + \varepsilon h_1, g + \varepsilon h_2)\Big|_{\varepsilon=0} = 0$$

اکنون به کمک خاصیت خطی بودن عملگر مشتق کسری کاپوتو و قاعده‌ی انتگرال‌گیری جزء به جزء [۲] برای حساب کسری و قضیه اساسی حساب تغییرات به معادلات ۱.۲ می‌رسیم.
□

در ادامه با کمک نامساوی پوانکاره در حساب کسری [۱] نشان می‌دهیم که تابعی ۱.۱ کوریسیو می‌باشد. سپس به کمک آن و خاصیت چند محدبی تابع زیر انتگرال در ضابطه‌ی $\mathbb{E}$ وجود منیمم‌کننده‌ها را ثابت می‌کنیم

لم ۲.۲. تابعی $\mathbb{E}$ روی فضای توابع $\mathfrak{B}$ از پایین کراندار است و ثابت $l = l(a, b, \alpha)$ وجود دارد، به طوری که

$$\mathbb{E}(\rho, g) \geq l[\parallel g \parallel_{L^2}^2 + (\parallel \rho(r) \parallel_{L^2} + a(b-a)^{\frac{1}{2}})^2]$$

برهان. با توجه به اینکه برای هر $(\rho, g) \in \mathfrak{B}$ روی بازه‌ی $[a,b]$ تابع $a \leq \rho \leq b$ $r \in [a,b]$، همچنین تابع $\Phi \geq 0$ و با کمک نامساوی پوانکاره داریم

$$\mathbb{E}(\rho, g) \geq \frac{(\Gamma(\alpha)^2)(2\alpha - 1)}{(b-a)^{2\alpha}}[\parallel g \parallel_{L^2}^2 + (\parallel \rho(r) \parallel_{L^2} + a(b-a)^{\frac{1}{2}})^2].$$

لذا با انتخاب $l = \frac{(\Gamma(\alpha)^2)(2\alpha-1)}{(b-a)^{2\alpha}}$ لم اثبات می‌شود. □

فضای توابع $\mathfrak{B}$ زیرفضایی بسته از فضای سوبولوف $W^{1,2}(a,b)$ است، بنابراین با توجه به قضیه نشاندن سوبولوف، می‌توان گفت کلاس‌های هموتوپی معرفی شده در مرجع [۳] فضای $\mathfrak{B}$ را نیز افراز می‌کنند. این کلاس‌ها را با نماد $c_m[\mathfrak{B}]$ برای $m \in \mathbb{Z}$ نشان می‌دهیم.

قضیه ۳.۲. تابعی انرژی $\mathbb{E}$ را روی فضای توابع $\mathfrak{B}$ در نظر بگیرید، در این صورت برای هر$m \in \mathbb{Z}$ زوج $(\rho, g) \in c_m[\mathfrak{B}]$ موجود است، به طوری که $\mathbb{E}$ را روی این کلاس هموتوپی مینیمم می‌کنند.

برهان. فرض کنید (ρ_n, g_n) دنباله‌ی مینیم‌کننده دلخواه $\mathbb{E}$ روی کلاس هموتوپی $c_m[\mathfrak{B}]$ باشد. بنابر لم ۲.۲ دارای زیر دنباله‌ی همگرا می‌باشد، بدون کم شدن از کلیت، دنباله مورد نظر را همین زیردنباله می‌گیریم، لذا داریم

$$\begin{cases} \rho_n \rightharpoonup \rho \quad \text{in} \quad W^{1,2}(a,b) \\ g_n \rightharpoonup g \quad \text{in} \quad W^{1,2}(a,b) \end{cases}$$

با توجه به خطی بودن مشتق کسری کاپوتو و کرانداری تابع ρ

$$\frac{\rho_n}{r}{}_a^C\mathcal{D}_r^\alpha \rho_n \rightharpoonup \frac{\rho}{r}{}_a^C\mathcal{D}_r^\alpha \rho \;\; in \;\; L^2(a,b)$$
$$\rho_n{}_a^C\mathcal{D}_r^\alpha g_n \rightharpoonup \rho_a^C\mathcal{D}_r^\alpha g \;\; in \;\; L^2(a,b)$$

از اینکه تابعی $\mathbb{E}$ دارای ویژگی به طور ضعیف پیوسته‌ی پایینی نیز می‌باشد، بنابراین

$$\begin{aligned} \inf_{c_m[\mathcal{B}]} \mathbb{E} &\leq \mathbb{E}(\rho, g) \\ &\leq \liminf_{n\to\infty} \mathbb{E}(\rho_n, g_n) \\ &\leq \inf_{c_m[\mathcal{B}]} \mathbb{E} \end{aligned}$$

کلاس‌های هموتوپی زیرفضاهای به طور ضعیف بسته از از فضای سوبولوف $W^{1,2}[a,b]$ می‌باشند، بنابراین $(\rho, g) \in c_m[\mathfrak{B}]$. $\square$

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قزوین، ایران

سخنرانی		*:سخنران

ساختار نگهدارنده های خطی وارون روابط مهتری

عرفان یزدان * و علی آرمندنژاد

گروه ریاضی، دانشکده علوم ریاضی، دانشگاه ولیعصر (عج) رفسنجان

چکیده. در این مقاله قصد داریم مفهوم نگهدارنده های خطی وارون را برای روابط مهتری، مهتری ضعیف و مهتری سطری مورد مطالعه و بررسی قرار دهیم و پیکربندی این نگهدارنده های خطی وارون را برای هرکدام از این روابط بدست آوریم.

۱. پیش‌گفتار

تعریف ۱.۱. [۱] فرض کنیم $M_{n,m}$ مجموعه تمام ماتریس های $n \times m$ با درایه های حقیقی باشد. برای هر $A, B \in M_{nm}$ گوییم B مهتر A است و با نماد $A \prec B$ نشان می دهیم هرگاه $A = DB$ که در آن $D \in M_{nn}$ یک ماتریس تصادفی دوگانه است.

تعریف ۲.۱. اگر $A, B \in M_{nm}$ باشند، آنگاه B دارای مهتری جهت دار نسبت به A است و می نویسیم $A \prec_d B$ ،هرگاه به ازای هر $x \in \mathbb{R}^n$ داشته باشیم $Ax \prec Ay$.

تعریف ۳.۱. [۲] در تعریف ۱.۱ اگر ماتریس تصادفی دوگانه D را با ماتریس سطری تصادفی $R \in M_{nn}$ جایگزین کنیم، گوییم B مهتر سطری A است و با نماد $A \prec_r B$ نشان می دهیم.

تعریف ۴.۱. [۱] اگر $x, y \in \mathbb{R}^n$ دو بردار دلخواه باشند، گوییم y مهتر ضعیف x است و با نماد $x \prec_w y$ نشان می دهیم هرگاه:

$$\sum_{i=1}^{n} x_i^{\downarrow} \leq \sum_{i=1}^{n} y_i^{\downarrow}$$

که در آن $x_i^{\downarrow}$ و $y_i^{\downarrow}$ به ترتیب آرایش نزولی درایه های بردارهای x و y می باشند.

واژگان کلیدی. مهتری، مهتری ضعیف، مهتری سطری، نگهدارنده خطی وارون .

* سخنران

تبدیل خطی $T : M_{nm} \longrightarrow M_{nm}$ را یک نگهدارنده خطی رابطه مهتری گویند هرگاه T دارای ویژگی زیر باشد:

به ازای هر $A, B \in M_{nm}$ اگر $A \prec B$ آنگاه $T(A) \prec T(B)$.

ساختار نگهدارنده های خطی رابطه های مهتری ذکر شده در بالا در [۲] و [۳] مشخص شده اند. در این مقاله ما مفهوم جدید عملگر خطی نگهدارنده وارون را بصورت زیر معرفی نموده و ساختار این عملگرهای خطی را در حالتهای مختلف تعیین می نماییم.

تعریف ۵.۱. عملگر خطی $T : M_{nm} \longrightarrow M_{nm}$ را نگهدارنده خطی وارون رابطه مهتری می گوییم هرگاه T دارای ویژگی زیر باشد:

به ازای هر $A, B \in M_{nm}$ اگر $A \prec B$ آنگاه $T(B) \prec T(A)$.

۲. دستآوردهای پژوهش

ابتدا ساختار نگهدارنده خطی وارون را برای بردارهای $\mathbb{R}^n$ تعیین می نماییم.

قضیه ۱.۲. فرض کنیم $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ یک عملگر خطی باشد بطوریکه $x \prec y$ نتیجه دهد $T(y) \prec T(x)$. آنگاه بردار $a \in \mathbb{R}^n$ یافت می شود بطوریکه:

$$T(x) = trx.a \quad , \quad \forall x \in \mathbb{R}^n.$$

برهان. طبق تعریف رابطه مهتری $\prec$ برای بردارهای $\mathbb{R}^n$، اگر e_i و e_j دو عضو دلخواه پایه استاندارد برای $\mathbb{R}^n$ باشند، آنگاه $0 \prec (e_i - e_j)$. با توجه به فرض قضیه داریم:

$$\begin{aligned} 0 \prec (e_i - e_j) &\Rightarrow T(e_i) - T(e_j) \prec 0 \\ &\Rightarrow T(e_i) - T(e_j) = 0 \\ &\Rightarrow T(e_i) = T(e_j) \end{aligned}$$

بنابراین اگر $[T]$ ماتریس متناظر با عملگر T باشد آنگاه دارای ستونهای یکسان است و این یعنی بردار $a \in \mathbb{R}^n$ یافت می شود بطوریکه:

$$[T] = [a\ a \cdots a] \Rightarrow T(x) = trx.a$$

بالعکس اگر $x \prec y$ و $T(x) = trx.a$ ، آنگاه داریم:

$$\begin{aligned} & x \prec y \Rightarrow trx = try \\ \Rightarrow\ & T(x) = trx.a = try.a = T(y) \\ \Rightarrow\ & T(y) \prec T(x). \end{aligned}$$

$\square$

اکنون با بهره گیری از نتیجه اصلی قضیه فوق و تعریف ۵.۱ و همچنین شیوه ای مشابه اثبات قضیه ۲ در مقاله [۳] ، قصد داریم فرم عملگر خطی نگهدارنده وارون T را برای ماتریس های M_{nm} تعیین نماییم. در اثبات قضیه زیر ما از این موضوع استفاده می کنیم که اگر $A \prec B$ آنگاه $A \prec_d B$ است.

قضیه ۲.۲. فرض کنید $T : M_{nm} \longrightarrow M_{nm}$ یک عملگر خطی باشد. آنگاه موارد زیر باهم معادلند:
الف) T نگهدارنده وارون مهتری جهت دار است.
ب) T نگهدارنده وارون مهتری است.
ج) $T(X) \prec_d T(Y)$ زمانی که $Y \prec X$.
د) اگر x_j ستون j ام ماتریس X باشد، ماتریس های $A_1, \dots, A_n$ وجود دارند بطوریکه $T(X) = \sum_{j=1}^{n} (trx_j) A_j$.

برهان. ابتدا نگاشت $E_j : \mathbb{R}^n \longrightarrow M_{nm}$ که $E_j(x) = xe_j^t$ برای $j = 1, \dots, m$ را تعریف می کنیم و سپس قرار می دهیم:

$$T_i^j = E_i^* T E_j$$

و ماتریس X را به فرم ستونی $X = \left[x_1 | \cdots | x_m \right]$ می نویسیم. آنگاه داریم:

$$T(x_1 | \cdots | x_m) = \left[\sum_{j=1}^{m} T_1^j x_j | \cdots | \sum_{j=1}^{m} T_m^j x_j \right]$$

اگر قسمت (ج) قضیه برقرار باشد، داریم:

$$x \prec y \Rightarrow T_i^j x \succ T_i^j y \qquad (۱.۲)$$

و از قضیه ۱.۲ نتیجه می شود که باید $T_i^j(x) = trx.a_i^j$. اکنون با قرار دادن $A_j = \left[a_1^j | \cdots | a_m^j \right]$ داریم:

$$\begin{aligned}
T(x_1 | \cdots | x_m) &= \left[\sum_{j=1}^{m} T_1^j x_j | \cdots | \sum_{j=1}^{m} T_m^j x_j \right] \\
&= \left[\sum_{j=1}^{m} (trx_j) a_1^j | \cdots | \sum_{j=1}^{m} (trx_j) a_m^j \right] \\
&= \sum_{j=1}^{m} (trx_j)(a_1^j | \cdots | a_m^j) \\
&= \sum_{j=1}^{m} (trx_j) A_j.
\end{aligned}$$

$\square$

در اثبات فوق ما از برقراری قسمت (ج) به حکم (د) رسیدیم. از طرفی اثبات ((د)⇐(الف)⇐(ج)) و((د)⇐(ب)⇐(ج)) واضح است و قضیه فوق بصورت کامل اثبات می گردد.

در قضیه بعد فرم عملگر خطی نگهدارنده وارون مهتری ضعیف را تعیین می کنیم.

قضیه ۳.۲. فرض کنید $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ یک عملگر خطی باشد بطوریکه $x \prec_w y$ نتیجه دهد $T(x) \prec_w T(y)$، آنگاه بردار $a \in \mathbb{R}^n$ با درایه های نامثبت وجود دارد بطوریکه $T(x) = trx.a$.

برهان. از آنجا که $(e_i - e_j) \prec_w 0$ ، طبق فرض قضیه داریم:

$$\begin{aligned} & T(e_i - e_j) \prec_w 0 \\ \Rightarrow \quad & T(e_i) - T(e_j) \prec_w 0 \end{aligned}$$

واگر $[T] = [t_{ij}]$ ، آنگاه:

$$\begin{bmatrix} t_{1i} - t_{1j} \\ \vdots \\ t_{ni} - t_{nj} \end{bmatrix} \prec_w 0$$

با توجه به تعریف مهتری ضعیف باید ماکسیمم بردار سمت چپ در عبارت فوق و در نتیجه تمام درایه های این بردار کوچکتر یا مساوی با صفر باشند. و با استدلالی مشابه برای $e_j - e_i$ و نتیجه ای که حاصل می شود، داریم:

$$t_{1i} - t_{1j} = \cdots = t_{ni} - t_{nj} = 0.$$

و این یعنی ستون های $[T]$ باهم برابرند. از طرفی چون $e_i \prec_w 0$ و $T(e_i) \prec_w 0$ ، پس درایه های ستونهای $[T]$ باید کوچکتر یا مساوی صفر باشند. $\square$

سرانجام در قضیه بعد فرم عملگر خطی نگهدارنده وارون برای رابطه مهتری سطری ارائه می شود.

قضیه ۴.۲. فرض کنید $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ یک نگاشت خطی باشد بطوریکه $x \prec_r y$ نتیجه دهد $T(y) \prec_r T(x)$ آنگاه $T = 0$.

برهان. اولا اگر $T = 0$ آنگاه حکم برگشت قضیه به وضوح برقرار است.
بالعکس. برای هر i، $e_i \prec_r 0$. طبق فرض قضیه داریم:

$$T(e_i) \prec_r 0 \Rightarrow T(e_i) = 0$$

و بنابراین عملگر خطی T برابر با صفر است. $\square$

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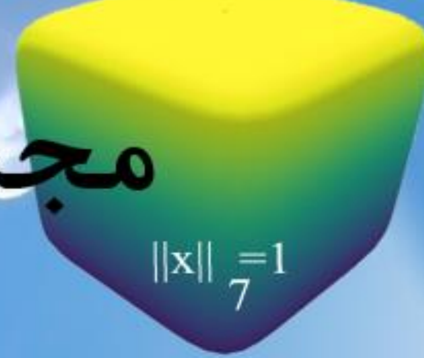

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